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# Manifestly covariant canonical quantization II: Gauge theory and anomalies

T. A. Larsson
Vanadisvägen 29, S-113 23 Stockholm, Sweden
email: thomas.larsson@hdd.se

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**Abstract**

In hep-th/0411028 a new manifestly covariant canonical quantization method was developed. The idea is to quantize in the phase space of arbitrary histories first, and impose dynamics as first-class constraints afterwards. The Hamiltonian is defined covariantly as the generator of rigid translations of the fields relative to the observer. This formalism is now applied to theories with gauge symmetries, in particular electromagnetism and Yang-Mills theory. The gauge algebra acquires an abelian extension proportional to the quadratic Casimir operator. Unlike conventional gauge anomalies proportional to the third Casimir, this is not inconsistent. On the contrary, a well-defined and non-zero charge operator is only compatible with unitarity in the presence of such anomalies. This anomaly is invisible in field theory because it is a functional of the observer's trajectory, which is conventionally ignored.

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*Dedicated to the memory of Julia Tengå and her parents, who disappeared in the Wave at Khao Lak.*

# 1 Introduction

In the first paper in this series [12], we introduced a new canonical quantization method, which preserves manifest covariance. It is based on two key ideas: regard the Euler-Lagrange (EL) equations as first-class constraints in the phase space of arbitary histories, and expand all fields in a Taylor series around the observer's trajectory in spacetime. The real motivation for introducing a new way to look at the old quantum theory becomes apparent in the present paper, where the formalism is applied to systems with constraints and gauge symmetries. *We demand that the constraint algebra be realized as well-defined operators on the kinematical Hilbert space.* A first step in this direction is to understand the quantum representation theory of the constraint algebra. Fortunately, in the last decade much has been learnt about the representations of algebras of diffeomorphism and gauge transformations in more than one dimension [2, 9, 10, 11, 16]. The time is now ripe to apply these insights to physics.

It is known but not always appreciated that the notion of gauge invariance is problematic on the quantum level already in QED. E.g., a recent article [19] begins: "This article is concerned with a major unsolved problem of QED, that of an exact formulation of gauge invariance, more especially the problem of an exact characterization of gauge transformations and their uses." The problem with gauge transformations on the quantum level is due to a real obstruction: in all unitary lowest-energy representations, the groups of gauge transformations and diffeomorphisms acquire quantum corrections. This is well known in one dimension, where the relevant algebras are the affine Kac-Moody and Virasoro algebras, respectively, but the same thing is true in higher dimensions as well.

Let us briefly outline the main ideas in this paper. Phase space is a covariant concept; it is the space of histories which solve the dynamics. Each phase space point $(q, p)$ generates a unique history $(q(t), p(t))$ under Hamiltonian evolution. We may choose to coordinatize phase space by $(q, p) = (q(0), p(0))$, but this is only a choice of coordinates, and physics is of course independent of this choice. We make the space of arbitrary histories $(q(t), p(t))$ into a phase space $\mathcal{P}$ by defining the Poisson brackets

$$[p(t), q(t')] = \delta(t - t'), \qquad [p(t), p(t')] = [q(t), q(t')] = 0. \tag{1.1}$$

The EL equations now define a constraint $\mathcal{E}(t) \approx 0$ in $\mathcal{P}$; since $\mathcal{E}(t)$ only depends on $q(t)$ this constraint is first class. This observation allows us to apply powerful cohomological methods from BRST quantization of theories with first class constraints. In other words, the idea is to quantize in the history phase space first and to impose dynamics afterwards, by passing to cohomology. Since dynamics is regarded as a constraint, this is non-trivial even for systems without gauge symmetries, like the harmonic oscillator and the free scalar field treated in [12].

The second idea is to expand all fields in a Taylor series around the observer's trajectory $q^\mu(t)$. Since covariance only makes sense for field theories, we now denote the histories by $(\phi(x), \pi(x))$, and define

$$\phi(x) = \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} \phi_{,\mathbf{m}}(t)(x - q(t))^{\mathbf{m}}. \tag{1.2}$$

One reason for describing the history phase space with Taylor coordinates $(\phi_{,\mathbf{m}}(t), q^\mu(t), \pi^{,\mathbf{m}}(t), p_\mu(t))$ rather than field coordinates $(\phi(x), \pi(x))$ is that we can write down a covariant expression for the Hamiltonian; the operator

$$H = i \int dt \, {:}\dot{q}^\mu(t) p_\mu(t){:} \tag{1.3}$$

translates the observer relative to the fields or vice versa. In particular, if $q^\mu(t) = u^\mu t$ where $u^\mu = (1, 0, 0, 0)$, then the Hamiltonian acts on the fields as expected:

$$[H, \phi(x)] = -i \frac{\partial}{\partial x^0} \phi(x). \tag{1.4}$$

Taken together, these two ideas yield a formulation of quantum theory which is both canonical and covariant. This is aesthetetically pleasing, but this is not really a good enough reason to motivate a new formalism. After all, decades of work have gone into path integrals and conventional canonical quantization, and those methods yield accurate predictions for observed quantities. In contrast, the only things that have been calculated so far with the present method are the spectra of the harmonic oscillator and the free scalar field [12]. The true motivation is the possibility to realize the gauge generators as well-defined operators.

Let us define what it means for an operator to be well-defined. Ideally, we want unitary and perhaps bounded operators acting on a Hilbert space. However, nothing is said about unitarity or the Hilbert space norm in this paper. Instead, we use the weaker notion that an operator acting on a

linear space with countable basis is well-defined if it takes a basis state into a *finite* linear combination of basis states. In particular, the brackets of two operators $[A, B]$ must itself be an operator, and an infinite constant times the unit operator is not well-defined.

In canonical quantization of low-dimensional systems, the constraint algebra is realized as well-defined operators on the kinematical Hilbert space. This is satisfied e.g. in first-quantized string theory, where the constraint algebra of Weyl rescalings is realized as the Virasoro operators

$$L_m = \sum_n :a_n a_{m-n}:, \tag{1.5}$$

which act on the kinematical Hilbert space generated by $a_m$'s with $m \geqslant 0$. However, in conventional canonical quantization of higher-dimensional theories such as Yang-Mills theory and general relativity in four dimensions, the quantum constraints are not well-defined operators, because infinities arise which can not be removed by normal ordering.

The relevant constraint algebras can typically be identified as algebras of gauge transformations or diffeomorphisms in four dimensions, whose natural habitat is a linear space built from a truncation of the history variables. More precisely, the space of Taylor coefficients with multi-indices of length $|\mathbf{m}| \leqslant p$ carries a nonlinear realization of these algebras. After quantization, we obtain a lowest-energy representation on a linear space. The representation is projective and has an appropriate concept of energy which is bounded from below. By a projective representation we mean that the constraint algebra acquires an extension, i.e. it becomes a higher-dimensional generalization of the affine Kac-Moody or the Virasoro algebras. It is hardly surprising that out formalism yields well-defined representations of such extended algebras, since it grew out of the representation theory of toroidal Lie algebras.

An extension of a gauge algebra can be considered as a gauge anomaly, which usually is considered to be inconsistent [14]. However, this is a far too simplistic point of view. A gauge symmetry is a redundancy of the description, and an anomaly therefore means that some gauge degrees of freedom become physical at the quantum level (quantum-mechanical gauge symmetry breaking). The anomalous theory *may* be inconsistent, but does not have to be so provided that the anomalous representation of the gauge algebra is unitary. If there is an anomaly, the gauge symmetry is of course no longer gauge, but must be treated as an anomalous global symmetry. Moreover, without a gauge anomaly it is really impossible to have non-zero electric charge, because the charge operator is a gauge generator. Conventionally,

this problem is ignored because gauge generators are not well-defined operators.

Since this view of gauge anomalies is unorthodox, Section 2 is devoted to a longer discussion of this topic, together with a general description of the BRST approach to quantization. That section also contains an explicit description of the Lie algebra extensions on the $N$-dimensional torus, making the connection to affine and Virasoro algebras very clear. It should be emphasized that the anomalies encountered here are of a new type, not visible in field theory, because the relevant cocycles are functionals of the observer's trajectory. Another important property is that they do not ruin the Noether identities; charge conservation is implemented as an operator equation, as explained in Section 7.

The philosophy in this paper is in many ways similar to the History Projection Operator formalism developed by Savvidou and others. In particular, the Poisson brackets (1.1) in the history phase space were first introduced in [17]. It was emphasized in [18] that conventional canonical quantization leads to severe conceptual difficulties when applied to covariant theories, especially gravity. A substantial difference is that the observer's trajectory was not explicitly introduced in those papers, and hence no observer-dependent anomalies were found.

The rest of the paper is organized as follows. Section 3 contains a general description of the cohomological approach to quantization. We introduce antifields and ghosts and obtain the reduced phase space in the BRST cohomology. However, the gauge generators and the BRST operator become ill defined after quantization. To remedy that, we regularize the theory in Section 4 by passing to $p$-jet space. The unique feature with this regularization is that the full gauge symmetry is manifest in the truncated theory. After quantization, the gauge generators remain well defined but the brackets may be anomalous, i.e. the BRST operator typically fails to be nilpotent. The BRST operator is a sum of two terms, $Q_{BRST} = Q_{KT} + Q_{Long}$, and the Koszul-Tate (KT) part $Q_{KT}$ is still nilpotent. This means that we must use the KT cohomology rather than the BRST cohomology to construct the physical Hilbert space. In Section 5 we describe the quantum (normal-ordered) form of the gauge generators and their associated anomalies.

In Section 6 the formalism is applied to the free Maxwell field. This case is special because it is anomaly free; the Kac-Moody-like cocycle is proportional to the second Casimir, which vanishes in the adjoint representation of $\mathfrak{u}(1)$. Standard results are recovered, in particular photons with two physical polarizations. There is some spurious cohomology because the Hessian is singular, but this phenomenon occurred already for the harmonic oscillator

[12] and has nothing to do with gauge invariance.

In the next section we couple a charged Dirac field to the Maxwell field. The main novelty here is the presence of a non-zero gauge anomaly. The KT operator still ensures that only the gauge-invariant field strength $F_{\mu\nu}$ survives in cohomology, but now there are three photon polarizations; in the interacting theory the photons are virtual, and virtual photons may have unphysical polarizations.

In Section 8 a pure non-abelian Yang-Mills field is considered. Here the second Casimir is non-zero already without fermions, because the gluons themselves carry color charge, and accordingly the Yang-Mills algebra acquires quantum corrections. The final section contains a brief conclusion.

## 2 Nilpotency, anomalies, and consistency

The cleanest way to deal with gauge symmetries is through the BRST formalism. For each gauge symmetry generator $J_a$, we introduce a ghost $c^a$ and ghost momentum $b_a$, and define the BRST operator

$$Q = J_a c^a - \frac{1}{2} f_{ab}{}^c c^a c^b b_c. \tag{2.1}$$

The classical BRST operator is nilpotent, $Q^2 = \{Q, Q\} = 0$, where $\{\cdot, \cdot\}$ is the symmetric bracket. A classical observable $\mathcal{O}$ is BRST closed, $[Q, \mathcal{O}] = 0$, and two observables are equivalent if they differ by a BRST exact term, $\mathcal{O} \sim \mathcal{O} + [Q, F]$. In other words, the space of functions on the extended phase space, including ghosts and ghost momenta, decomposes into subspaces of fixed ghost number, $C(\mathcal{P}^*) = \sum_k C^k$, such that

$$\dots C^{-2} \xrightarrow{Q} C^{-1} \xrightarrow{Q} C^0 \xrightarrow{Q} C^1 \xrightarrow{Q} C^2 \xrightarrow{Q} \dots . \tag{2.2}$$

The space of classical observables can be identified with the zeroth cohomology group of this complex,

$$H^0(Q) = \frac{(\ker Q)_0}{(\mathrm{im}\ Q)_0}, \tag{2.3}$$

whereas the higher cohomology groups are zero; we say that the complex (2.2) gives a resolution of the physical (gauge-invariant) phase space.

After quantization, we obtain two cohomologies. In the state cohomology, a physical state is BRST closed, $Q\big|phys\big\rangle = 0$, and two states are equivalent if they differ by an exact term, $\big|phys\big\rangle \sim \big|phys\big\rangle + Q\big|\big\rangle$. In the operator cohomology, a physical operator is BRST closed, $[Q, \mathcal{O}] = 0$, and two

operators are equivalent if they differ by a BRST exact term, $\mathcal{O} \sim \mathcal{O} + [Q, F]$. It is a standard result that these definitions are compatible.

One thing may go wrong upon quantization: the gauge symmetry may acquire quantum corrections, i.e. an anomaly. The gauge algebra then assumes the form

$$[J_a, J_b] = f_{ab}{}^c J_c + D_{ab}, \tag{2.4}$$

and the quantum BRST operator ceases to be nilpotent, $\{Q, Q\} \sim D_{ab}$. Then it makes no sense to require that $Q\big|phys\big\rangle = 0$, because it would mean that

$$\{Q, Q\}\big|phys\big\rangle \sim D_{ab}\big|phys\big\rangle = 0. \tag{2.5}$$

In particular, if $D_{ab}$ is an invertible operator this would mean that $\big|phys\big\rangle = 0$, i.e. that there are no physical states at all.

However, it must be realized that the extension (2.4) is not necessarily inconsistent by itself. The inconsistency was our requirement that all physical states be annihilated by the no longer nilpotent BRST operator. It is quite conceivable that the unreduced Hilbert space has a positive-definite norm which is preserved by the algebra (2.4) and by time evolution. This happens in the chiral Schwinger model [6] and perhaps also in Liouville field theory. More importantly, anomalous conformal symmetry plays an important role in two-dimensional statistical physics [3]. In fact, it is well known in this context that infinite-dimensional spacetime symmetries (gauge or not) are compatible with locality, in the sense of correlation functions which depend on separation, only in the presence of an anomaly.

In the situation that we are interested in, the BRST operator can be written as $Q \equiv Q_{BRST} = Q_{KT} + Q_{Long}$, where the Koszul-Tate (KT) operator $Q_{KT}$ always remains nilpotent after quantization, but the longitudinal part $Q_{Long}$ may become anomalous. Then any state of the form $Q_{KT}\big|\big\rangle$ can and must be modded out. This is done by identifying the physical Hilbert space with the zeroth KT cohomology group. There are thus two different situations:

1. $Q^2_{BRST} = 0$. Then $\mathcal{H}_{phys} = H^0(Q_{BRST})$, and the gauge degrees of freedom are truly redundant, even after quantization.

2. $Q^2_{BRST} \neq 0$ but $Q^2_{KT} = 0$. Then $\mathcal{H}_{phys} = H^0(Q_{KT})$, and the gauge degrees of freedom become physical after quantization.

In either case one must check that the final theory is consistent, i.e. that $\mathcal{H}_{phys}$ has a positive-definite norm which is preserved by time evolution and all symmetries. This condition may or may not fail.

It is common to instinctively reject all gauge anomalies as inconsistent, due to experience with the standard model. It is therefore important to point out that the gauge anomalies considered in the present paper are of a different type than conventional ones. In field theory, anomalies typically arise when chiral fermions are coupled to gauge fields, and they are proportional to the third Casimir operator [1, 14]. In contrast, the gauge anomalies described here are functionals of the observer's trajectory in spacetime, and they are proportional to the second Casimir. In particular, they only vanish for the pure Maxwell field. Moreover, within field theory no gravitational anomalies at all exist in four dimensions, but a generalization of the Virasoro algebra certainly exists in any number of dimensions, and this extension arises upon quantization.

To make contact with the Virasoro algebra in its most familiar form, we describe its multi-dimensional sibling in a Fourier basis on the $N$-dimensional torus. Recall first that the algebra of diffeomorphisms on the circle, $\mathfrak{vect}(1)$, has generators

$$L_m = -i \exp(imx) \frac{d}{dx}, \tag{2.6}$$

where $x \in S^1$. $\mathfrak{vect}(1)$ has a central extension, known as the Virasoro algebra:

$$[L_m, L_n] = (n-m) L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}, \tag{2.7}$$

where $c$ is a c-number known as the central charge or conformal anomaly. This means that the Virasoro algebra is a Lie algebra; anti-symmetry and the Jacobi identities still hold. The term linear in $m$ is unimportant, because it can be removed by a redefinition of $L_0$. The cubic term $m^3$ is a non-trivial extension which cannot be removed by any redefinition.

The generators (2.6) immediately generalize to vector fields on the $N$-dimensional torus:

$$L_\mu(m) = -i \exp(im_\rho x^\rho) \partial_\mu, \tag{2.8}$$

where $x = (x^\mu)$, $\mu = 1, 2, ..., N$ is a point in $N$-dimensional space and $m = (m_\mu)$. The Einstein convention is used; repeated indices, one up and one down, are implicitly summed over. These operators generate the algebra $\mathfrak{vect}(N)$:

$$[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n). \tag{2.9}$$

The question is now whether the Virasoro extension, i.e. the $m^3$ term in (2.7), also generalizes to higher dimensions.

Rewrite the ordinary Virasoro algebra (2.7) as

$$\begin{aligned}
[L_m, L_n] &= (n-m)L_{m+n} + cm^2 n S_{m+n}, \\
[L_m, S_n] &= (n+m)S_{m+n}, \\
[S_m, S_n] &= 0, \\
mS_m &= 0.
\end{aligned} \tag{2.10}$$

It is easy to see that the two formulations of $Vir$ are equivalent (I have absorbed the linear cocycle into a redefinition of $L_0$). The second formulation immediately generalizes to $N$ dimensions. The defining relations are

$$\begin{aligned}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n) \\
&\quad +(c_1 m_\nu n_\mu + c_2 m_\mu n_\nu) m_\rho S^\rho(m+n), \\
[L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m+n) + \delta^\nu_\mu m_\rho S^\rho(m+n), \\
[S^\mu(m), S^\nu(n)] &= 0, \\
m_\mu S^\mu(m) &= 0.
\end{aligned} \tag{2.11}$$

This is an extension of $\mathfrak{vect}(N)$ by the abelian ideal with basis $S^\mu(m)$. Geometrically, we can think of $L_\mu(m)$ as a vector field and $S^\mu(m) = \epsilon^{\mu\nu_2..\nu_N} S_{\nu_2..\nu_N}(m)$ as a dual one-form (and $S_{\nu_2..\nu_N}(m)$ as an $(N-1)$-form); the last condition expresses closedness. The cocycle proportional to $c_1$ was discovered by Rao and Moody [16], and the one proportional to $c_2$ by this author [8].

There is also a similar multi-dimensional generalization of affine Kac-Moody algebras, presumably first written down by Kassel [7]. It is sometimes called the central extension, but this term is somewhat misleading because the extension does not commute with diffeomorphisms, although it does commute with all gauge transformations.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with structure constants ${f_{ab}}^c$ and Killing metric $\delta_{ab}$. The Kassel extension of the current algebra $\mathfrak{map}(N, \mathfrak{g})$ is defined by the brackets

$$\begin{aligned}
[J_a(m), J_b(n)] &= {f_{ab}}^c J_c(m+n) + k\delta_{ab} m_\rho S^\rho(m+n), \\
[J_a(m), S^\mu(n)] &= [S^\mu(m), S^\nu(n)] = 0, \\
m_\mu S^\mu(m) &\equiv 0.
\end{aligned} \tag{2.12}$$

This algebra admits an intertwining action of $Vir(N)$:

$$[L_\mu(m), J_a(n)] = n_\mu J_a(m+n). \tag{2.13}$$

The current algebra $\mathfrak{map}(N, \mathfrak{g})$ also admits another type of extension in some dimensions. The best known example is the Mickelsson-Faddeev algebra, relevant for the conventional anomalies in field theory, which arise when chiral fermions are coupled to gauge fields in three spatial dimensions. Let $d_{abc} = \mathrm{tr}\{T_a, T_b\}T_c$ be the totally symmetric third Casimir operator, and let $\epsilon^{\mu\nu\rho}$ be the totally anti-symmetric epsilon tensor in three dimensions. The Mickelsson-Faddeev algebra [13] reads in a Fourier basis:

$$\begin{aligned}
[J_a(m), J_b(n)] &= f_{ab}{}^c J_c(m+n) + d_{abc}\epsilon^{\mu\nu\rho} m_\mu n_\nu A^c_\rho(m+n), \\
[J_a(m), A^b_\nu(n)] &= f_{ab}{}^c A^c_\nu(m+n) + \delta^b_a m_\nu \delta(m+n), \\
[A^a_\mu(m), A^b_\nu(n)] &= 0.
\end{aligned} \tag{2.14}$$

$A^a_\mu(m)$ are the Fourier components of the gauge connection.

Note that $Q_a \equiv J_a(0)$ generate a Lie algebra isomorphic to $\mathfrak{g}$, whose Cartan subalgebra is identified with the charges. Moreover, the subalgebra of (2.12) spanned by $J_a(m_0) \equiv J_a(m)$, where $m = (m_0, 0, ..., 0) \in \mathbb{Z}$, reads

$$[J_a(m_0), J_b(n_0)] = f_{ab}{}^c J_c(m_0+n_0) + k\delta_{ab} m_0 \delta(m_0+n_0), \tag{2.15}$$

which we recognize as the affine algebra $\widehat{\mathfrak{g}}$. Since all non-trivial unitary irreps of $\widehat{\mathfrak{g}}$ has $k > 0$ [4], it is impossible to combine unitary and non-zero $\mathfrak{g}$ charges also for the higher-dimensional algebra (2.12). This follows immediately from the fact that the restriction of a unitary irrep to a subalgebra is also unitary (albeit in general reducible). Hence, rather than being inconsistent, the anomaly in (2.12) is indeed necessary for consistently including charge. This problem is circumvented in field theory simply because e.g. the electric charge is not a well-defined operator [19].

In contrast, the Mickelsson-Faddeev algebra (2.14) has apparently no faithful unitary representations on a separable Hilbert space [15], which presumably means that that it should be avoided. Indeed, Nature appears to abhor this kind of anomaly, which is proportional to the third Casimir.

# 3 Koszul-Tate and BRST cohomologies for constrained systems

The Hamiltonian formulation of quantum theory is more fundamental than the Lagrangian one, but it breaks manifest covariance. This is a major disadvantage, especially for theories with local symmetries. A manifestly covariant Hamiltonian formulation in the absense of gauge symmetries was described in [12], inspired by the Batalin-Vilkovisky or antifield approach [5].

The idea is to consider the space of arbitrary histories and their momenta as a phase space, and dynamics, i.e. the Euler-Lagrange (EL) equations, as first class constraints. This enables us to apply BRST techniques in the history phase space; quantize first and impose dynamics afterwards. Let us review this idea and then extend it to theories with irreducible gauge symmetries.

Consider a dynamical system described by a set of fields $\phi^\alpha$ and an action $S$. The EL equations read

$$\mathcal{E}_\alpha = \partial_\alpha S \equiv \frac{\delta S}{\delta\phi^\alpha} = 0. \tag{3.1}$$

We introduce an antifield $\phi^*_\alpha$ for each EL equation $\mathcal{E}_\alpha = 0$, and replace the space of $\phi$-histories $\mathcal{Q}$ by the extended history space $\mathcal{Q}^*$, spanned by both $\phi$ and $\phi^*$. In $\mathcal{Q}^*$ we define the Koszul-Tate (KT) differential $\delta$ by

$$\begin{aligned} \delta\phi^\alpha &= 0, \\ \delta\phi^*_\alpha &= \mathcal{E}_\alpha. \end{aligned} \tag{3.2}$$

One checks that $\delta$ is nilpotent, $\delta^2 = 0$. Define the antifield number $\operatorname{afn}\phi^\alpha = 0$, $\operatorname{afn}\phi^*_\alpha = 1$. The KT differential clearly has antifield number $\operatorname{afn}\delta = -1$.

The function space $C(\mathcal{Q}^*)$ decomposes into subspaces $C^k(\mathcal{Q}^*)$ of fixed antifield number

$$C(\mathcal{Q}^*) = \sum_{k=0}^{\infty} C^k(\mathcal{Q}^*) \tag{3.3}$$

The KT differential makes $C(\mathcal{Q}^*)$ into a differential complex,

$$0 \xleftarrow{\delta} C^0 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^2 \xleftarrow{\delta} \ldots \tag{3.4}$$

The cohomology spaces are defined as usual by $H^\bullet_{cl}(\delta) = \ker\delta/\mathrm{im}\ \delta$, i.e. $H^k_{cl}(\delta) = (\ker\delta)_k/(\mathrm{im}\ \delta)_k$, where the subscript *cl* indicates that we deal with a classical phase space. It is easy to see that

$$\begin{aligned} (\ker\delta)_0 &= C(\mathcal{Q}), \\ (\mathrm{im}\ \delta)_0 &= C(\mathcal{Q})\mathcal{E}_\alpha \equiv \mathcal{N}. \end{aligned} \tag{3.5}$$

Thus $H^0_{cl}(\delta) = C(\mathcal{Q})/\mathcal{N} = C(\Sigma)$ is identified with the space of functions over physical phase space $\Sigma$. Since we assume that there are no non-trivial relations among the $\mathcal{E}_\alpha$, the higher cohomology groups vanish. This is a

standard result [5]. The complex (3.4) thus gives us a resolution of $C(\Sigma)$, which by definition means that $H^0_{cl}(\delta) = C(\Sigma)$, $H^k_{cl}(\delta) = 0$, for all $k > 0$.

A key observation in [12] was that the same space $C(\Sigma)$ admits a different resolution, under the technical assumption that the Hessian (the second functional-derivative matrix of the action) is invertible. Introduce canonical momenta $\pi_\alpha = \delta/\delta\phi^\alpha$ and $\pi^\alpha_* = \delta/\delta\phi^*_\alpha$ for both the fields and antifields. The momenta satisfy by definition the graded canonical commutation relations

$$\begin{aligned} [\pi_\beta, \phi^\alpha] &= \delta^\alpha_\beta, & [\phi^\alpha, \phi^\beta] &= [\pi_\alpha, \pi_\beta] = 0, \\ \{\pi^\beta_*, \phi^*_\alpha\} &= \delta^\beta_\alpha, & \{\phi^*_\alpha, \phi^*_\beta\} &= \{\pi^\alpha_*, \pi^\beta_*\} = 0. \end{aligned} \tag{3.6}$$

We denote by $\mathcal{P}$ the phase space of arbitrary histories, with basis $(\phi^\alpha, \pi_\beta)$, and by $\mathcal{P}^*$ the extended phase space with basis $(\phi^\alpha, \pi_\beta, \phi^*_\alpha, \pi^\beta_*)$.

The definition of the KT differential extends to $\mathcal{P}^*$ by requiring that $\delta F = [Q_{KT}, F]$ for every $F \in C(\mathcal{P}^*)$, where the KT operator is $Q_{KT} = \mathcal{E}_\alpha \pi^\alpha_*$. Let us prove that this formula indeed yields a nilpotent differential $\delta$ defined on all of $C(\mathcal{P}^*)$. For brevity, set $\phi^A = (\phi^\alpha, \phi^*_\alpha)$ and $\pi_A = (\pi_\alpha, \pi^\alpha_*)$. For any functional $F(\phi)$,

$$\begin{aligned} \delta^2[\pi_A, F] &= \delta([\delta\pi_A, F] \pm [\pi_A, \delta F] \\ &= [\delta^2\pi_A, F] \mp [\delta\pi_A, \delta F] \pm [\delta\pi_A, \delta F] + [\pi_A, \delta^2 F] \\ &= [\delta^2\pi_A, F] \\ &= \delta^2(\partial_A F) = 0, \end{aligned} \tag{3.7}$$

because $\partial_A F$ is a functional of $\phi$. Hence we conclude that $[\delta^2\pi_A, F(\phi)] = 0$ for every $F$. On the other hand, $Q_{KT}$ is linear in $\pi_A$, so there must exist some functions $f^B_A(\phi)$ such that $\delta^2\pi_A = f^B_A(\phi)\pi_B$. But since $f^B_A(\phi)\partial_B F(\phi) \equiv 0$ for all $F$, we conclude that $f^B_A$ must vanish themselves. Hence $\delta^2\pi_A = 0$, and the definition of $\delta$ extends to the momenta. QED.

Because of this observation, we will usually not write down the action of the KT differential on the momenta, but only on the fields and antifields. Moreover, the same result holds for any nilpotent operator which is linear in momenta, such as the longitudinal and BRST differential encountered below.

The new ingredient in the present paper is that we assume that there are some relations between the EL equations (3.1). In other words, let there be identities of the form

$$r^\alpha_a \mathcal{E}_\alpha \equiv 0, \tag{3.8}$$

where the $r_a^\alpha$ are some functionals of $\phi^\alpha$. The zeroth cohomology group $H^0_{cl}(\delta) = C(\mathcal{Q})/\mathcal{N} = C(\Sigma)$ is not changed, but the higher cohomology groups no longer vanish, since $\delta(r_a^\alpha \phi^*_\alpha) = r_a^\alpha \mathcal{E}_\alpha \equiv 0$. The standard method to kill this unwanted cohomology is to introduce a bosonic second-order antifield $\zeta_a$, so that $r_a^\alpha \phi^*_\alpha = \delta\zeta_a$ is KT exact. The differential (3.2) is thus modified to read

$$\begin{aligned} \delta\phi^\alpha &= 0, \\ \delta\phi^*_\alpha &= \mathcal{E}_\alpha, \\ \delta\zeta_a &= r_a^\alpha \phi^*_\alpha. \end{aligned} \tag{3.9}$$

By introducing canonical momenta $\chi^a = \delta/\delta\zeta_a$ for the second-order antifields, we can write the KT differential as a bracket, $\delta F = [Q_{KT}, F]$, where the full KT operator is

$$Q_{KT} = \mathcal{E}_\alpha \pi_*^\alpha + r_a^\alpha \phi^*_\alpha \chi^a. \tag{3.10}$$

$Q_{KT}$ is an operator in the extended phase space $\mathcal{P}^*$ with basis $(\phi^\alpha, \pi_\beta, \phi^*_\alpha, \pi_*^\beta, \zeta_a, \chi^b)$, and $\{Q_{KT}, Q_{KT}\} = 0$.

The identity (3.8) implies that $J_a = r_a^\alpha \pi_\alpha$ generate a Lie algebra under the Poisson bracket. Namely, all $J_a$'s preserve the action, because

$$[J_a, S] = r_a^\alpha [\pi_\alpha, S] = r_a^\alpha \mathcal{E}_\alpha \equiv 0, \tag{3.11}$$

and the bracket of two operators which preserve some structure also preserves the same structure. We will only consider the case that the $J_a$'s generate a proper Lie algebra $\mathfrak{g}$ with structure constants $f_{ab}{}^c$,

$$[J_a, J_b] = f_{ab}{}^c J_c. \tag{3.12}$$

The formalism extends without too much extra work to the more general case of structure functions $f_{ab}{}^c(\phi)$, but we will not need this complication in this paper. It follows that the functions $r_a^\alpha$ satisfy the identity

$$\partial_\beta r_b^\alpha r_a^\beta - \partial_\beta r_a^\alpha r_b^\beta = f_{ab}{}^c r_c^\alpha. \tag{3.13}$$

The Lie algebra $\mathfrak{g}$ also acts on the antifields:

$$\begin{aligned} [J_a, \phi^\alpha] &= r_a^\alpha, \\ [J_a, \phi^*_\alpha] &= -\partial_\alpha r_a^\beta \phi^*_\beta \\ [J_a, \zeta_b] &= f_{ab}{}^c \zeta_c. \end{aligned} \tag{3.14}$$

In particular, it follows that $\phi^*_\alpha$ carries a $\mathfrak{g}$ representation because it transforms in the same way as $\pi_\alpha$ does.

Classically, it is always possible to reduce the phase space further, by identifying points on $\mathfrak{g}$ orbits. To implement this additional reduction, we introduce ghosts $c^a$ with anti-field number $\mathrm{afn}\, c^a = -1$, and ghost momenta $b_a$ satisfying

$$\{b_a, c^b\} = \delta^b_a. \tag{3.15}$$

The Lie algebra $\mathfrak{g}$ acts on the ghosts as

$$[J_a, c^b] = -f_{ac}{}^b c^c. \tag{3.16}$$

The full extended phase space, still denoted by $\mathcal{P}^*$, is spanned by $(\phi^\alpha, \pi_\beta, \phi^*_\alpha, \pi^\beta_*, \zeta_a, \chi^b, c^a, b_b)$. The generators of $\mathfrak{g}$ are thus identified with the following vector fields in $\mathcal{P}^*$:

$$\begin{aligned} J_a &= r^\alpha_a \pi_\alpha - \partial_\alpha r^\beta_a \phi^*_\beta \pi^\alpha_* + f_{ab}{}^c \zeta_c \chi^b - f_{ab}{}^c c^b b_c \\ &= J^{field}_a + J^{ghost}_a, \end{aligned} \tag{3.17}$$

where $J^{ghost}_a = f_{ab}{}^c c^b b_c$ and $J^{field}_a$ is the rest.

Now define the longitudinal derivative $d$ by

$$\begin{aligned} dc^a &= -\frac{1}{2} f_{bc}{}^a c^b c^c, \\ d\phi^\alpha &= r^\alpha_a c^a, \\ d\phi^*_\alpha &= \partial_\alpha r^\beta_a \phi^*_\beta c^a, \\ d\zeta_a &= -f_{ab}{}^c \zeta_c c^b. \end{aligned} \tag{3.18}$$

The longitudinal derivative can be written as $dF = [Q_{Long}, F]$ for every $F \in C(\mathcal{Q}^*)$, where

$$Q_{Long} = J^{field}_a c^a - \frac{1}{2} f_{ab}{}^c c^a c^b b_c = J^{field}_a c^a + \frac{1}{2} J^{ghost}_a c^a. \tag{3.19}$$

One verifies that $d^2 = 0$ when acting on the fields and antifields by means of the identify (3.13) and the Jacobi identities for $\mathfrak{g}$. Moreover, it is straightforward to show that $d$ anticommutes with the KT differential, $d\delta = -\delta d$; the proof is again done by checking the action on the fields.

Hence we may define the nilpotent *BRST derivative* $s = \delta + d$,

$$\begin{aligned} sc^a &= -\frac{1}{2} f_{bc}{}^a c^b c^c, \\ s\phi^\alpha &= r_a^\alpha c^a, \\ s\phi^*_\alpha &= \mathcal{E}_\alpha + \partial_\alpha r_a^\beta \phi^*_\beta c^a, \\ s\zeta_a &= r_a^\alpha \phi^*_\alpha - f_{ab}{}^c \zeta_c c^b. \end{aligned} \tag{3.20}$$

Nilpotency immediately follows because $s^2 = \delta^2 + \delta d + d\delta + d^2 = 0$. The BRST operator can be written in the form $sF = [Q_{BRST}, F]$ with

$$\begin{aligned} Q_{BRST} &= Q_{KT} + Q_{Long} \\ &= \mathcal{E}_\alpha \pi^\alpha_* + r_a^\alpha \phi^*_\alpha \chi^a + J_a^{field} c^a + \frac{1}{2} J_a^{ghost} c^a \\ &= -\frac{1}{2} f_{ab}{}^c c^a c^b b_c + r_a^\alpha c^a \pi_\alpha + (\mathcal{E}_\alpha + \partial_\alpha r_a^\beta \phi^*_\beta c^a) \pi^\alpha_* \\ &\quad + (r_a^\alpha \phi^*_\alpha - f_{ab}{}^c \zeta_c c^b) \chi^a. \end{aligned} \tag{3.21}$$

Like $C(\mathcal{Q})$, the function space $C(\mathcal{P}^*)$ decomposes into subspaces of fixed antifield number, $C(\mathcal{P}^*) = \sum_{k=-\infty}^{\infty} C^k(\mathcal{P}^*)$. We can therefore define a BRST complex in $C(\mathcal{P}^*)$

$$\ldots \xleftarrow{s} C^{-2} \xleftarrow{s} C^{-1} \xleftarrow{s} C^0 \xleftarrow{s} C^1 \xleftarrow{s} C^2 \xleftarrow{s} \ldots \tag{3.22}$$

It is important that the spaces $C^k$ in (3.22) are phase spaces, equipped with the Poisson bracket (3.6). Unlike the resolution (3.4), the new resolution (3.22) therefore allows us to do canonical quantization already in $C(\mathcal{P}^*)$: replace Poisson brackets by commutators and represent the graded Heisenberg algebra on a Hilbert space. To pick the correct Hilbert space, we must define a Hamiltonian which is bounded on below. Note that different choices may be inequivalent, because there is no Stone-von Neumann theorem in infinite dimension.

In non-covariant quantization, we single out a privileged variable $t$ among the $\alpha$'s, and declare it to be time. We thus make the substitution $\phi^\alpha \to \phi^\alpha(t)$, $\phi^*_\alpha \to \phi^*_\alpha(t)$, $\zeta_a \to \zeta_a(t)$, $c^a \to c^a(t)$, $\mathcal{E}_\alpha \to \mathcal{E}_\alpha(t)$, and similar for the momenta. The constraints (3.8) take the form

$$\int dt'\ r_a^\alpha(t,t') \mathcal{E}_\alpha(t') \equiv 0 \tag{3.23}$$

and the Lie algebra (3.12) becomes

$$[J_a(t), J_b(t')] = \int dt''\ f_{ab}{}^c(t,t',t'') J_c(t''). \tag{3.24}$$

The BRST action on the fields (3.20) is replaced by

$$\begin{aligned}
sc^a(t) &= -\frac{1}{2}\iint dt'dt''\ f_{bc}{}^a(t,t',t'')c^b(t')c^c(t''), \\
s\phi^\alpha(t) &= \int dt'\ r_a^\alpha(t,t')c^a(t'), \\
s\phi^*_\alpha(t) &= \mathcal{E}_\alpha(t) + \iint dt'dt''\ \partial_\alpha r_a^\beta(t,t',t'')\phi^*_\beta(t')c^a(t''), \\
s\zeta_a(t) &= \int dt'\ r_a^\alpha(t,t')\phi^*_\alpha(t') - \iint dt'dt''\ f_{ab}{}^c(t,t',t'')\zeta_c(t')c^b(t''),
\end{aligned} \tag{3.25}$$

where we defined

$$\partial_\alpha r_a^\beta(t,t',t'') \equiv \frac{\delta r_a^\beta(t,t')}{\delta\phi^\alpha(t'')}. \tag{3.26}$$

The Hamiltonian in history phase space is the generator of rigid time translations, *viz.*

$$H = -i\int dt\ \Big(\dot{\phi^\alpha}(t)\pi_\alpha(t) + \dot{\phi^*_\alpha}(t)\pi_*^\alpha(t) + \dot{\zeta}_a(t)\chi^a(t) + \dot{c}^a(t)b_a(t)\Big). \tag{3.27}$$

The Hamiltonian acts on the fields as

$$\begin{aligned}
[H,\phi^\alpha(t)] &= -i\dot{\phi^\alpha}(t), & [H,\phi^*_\alpha(t)] &= -i\dot{\phi^*_\alpha}(t), \\
[H,\zeta_a(t)] &= -i\dot{\zeta}_a(t), & [H,c^a(t)] &= -i\dot{c}^a(t).
\end{aligned} \tag{3.28}$$

The action on the momenta follows similarly from (3.27).

Expand all fields in a Fourier integral with respect to time, e.g,

$$\phi^\alpha(t) = \int_{-\infty}^{\infty} dm\ \phi^\alpha(m)\mathrm{e}^{imt}. \tag{3.29}$$

The Hamiltonian acts on the Fourier modes as

$$\begin{aligned}
[H,\phi^\alpha(m)] &= m\phi^\alpha(m), & [H,\phi^*_\alpha(m)] &= m\phi^*_\alpha(m), \\
[H,\zeta_a(m)] &= m\zeta_a(m), & [H,c^a(m)] &= mc^a(m).
\end{aligned} \tag{3.30}$$

Now quantize. In the spirit of BRST quantization, our strategy is to quantize first and impose dynamics afterwards. In the extended history phase space $\mathcal{P}^*$, we define a Fock vacuum $\big|0\big\rangle$ which is annihilated by all negative frequency modes, i.e.

$$\begin{aligned}
\phi^\alpha(-m)\big|0\big\rangle = \phi^*_\alpha(-m)\big|0\big\rangle = \zeta_a(-m)\big|0\big\rangle = c^a(-m)\big|0\big\rangle = 0, \\
\pi_\alpha(-m)\big|0\big\rangle = \pi_*^\alpha(-m)\big|0\big\rangle = \chi^a(-m)\big|0\big\rangle = b_a(-m)\big|0\big\rangle = 0,
\end{aligned} \tag{3.31}$$

for all $-m < 0$. We must also decide which of the zero modes that annihilate the vacuum, but this decision will not affect the eigenvalues of the Hamiltonian.

The Hamiltonian (3.27) does not act in a well-defined manner, because it assigns an infinite energy to the Fock vacuum. To correct for that, we replace the Hamiltonian by

$$
\begin{aligned}
H &= -i \int dt \, \Big( :\dot{\phi}^\alpha(t) \pi_\alpha(t): + :\dot{\phi}^*_\alpha(t) \pi^\alpha_*(t): \\
&\quad + :\dot{\zeta}_a(t) \chi^a(t): + :\dot{c}^a(t) b_a(t): \Big),
\end{aligned}
\tag{3.32}
$$

where normal ordering $:\cdot:$ moves negative frequency modes to the right and positive frequency modes to the left. The vacuum has zero energy as measured by the normal-ordered Hamiltonian, $H\big|0\big\rangle = 0$. The history Hilbert space $\mathcal{H}(\mathcal{P}^*)$ can be identified with the space of functions of $\phi^\alpha(m)$, $\phi^*_\alpha(m)$, $\zeta_a(m)$, $c^a(m)$, and $\pi_\alpha(m)$, $\pi^\alpha_*(m)$, $\chi^a(m)$, $b_a(m)$, where all $m > 0$. The energy of a state $\phi^{a_1}(m_1) ... b_{a_n}(m_n)\big|0\big\rangle$ in $\mathcal{H}(\mathcal{P}^*)$ is simply the total frequency $(m_1 + ... + m_n)$.

In the absense of gauge symmetries, the BRST operator reduces to the KT operator (3.10), which is already normal ordered. Writing out the time dependence explicity, we have

$$
\begin{aligned}
Q_{KT} &= \int dt \ :\mathcal{E}_\alpha(t) \pi^\alpha_*(t): + \iint dt dt' \ :r^\alpha_a(t,t') \phi^*_\alpha(t) \chi^a(t'): \\
&= \int dt \ \mathcal{E}_\alpha(t) \pi^\alpha_*(t) + \iint dt dt' \ r^\alpha_a(t,t') \phi^*_\alpha(t) \chi^a(t'),
\end{aligned}
\tag{3.33}
$$

since $\mathcal{E}_\alpha(t)$ and $r^\alpha_a(t,t')$ depend on the fields $\phi^\alpha(t)$ only, and not on the antifields $\phi^*_\alpha(t)$ and $\zeta_a(t)$. This means that quantum effects do not spoil the nilpotency of $Q_{KT}$, so the KT complex is well defined on the quantum level. However, the BRST operator is not necessarily nilpotent. The dangerous part is the longitudinal operator

$$
Q_{Long} = \int dt \ \Big( :J^{field}_a(t): c^a(t) + \frac{1}{2} :J^{ghost}_a(t) c^a(t): \Big).
\tag{3.34}
$$

The longitudinal operator ceases to be nilpotent unless the normal-ordered gauge generators $J_a(t) = \, :J^{field}_a(t): + :J^{ghost}_a(t):$ generate the algebra (3.12) without additional quantum corrections. If so, the BRST operator also ceases to be nilpotent.

However, everything is not lost. The KT operator is still nilpotent, and we can implement dynamics as the KT cohomology in the extended phase

space without ghosts. The physical phase space now grows, because some gauge degrees of freedom become physical upon quantization. The classical gauge symmetry has now become an ordinary, "global" symmetry (albeit still local in spacetime), which must be realized as well-defined, unitary operators. This is a highly non-trivial requirement, which typically fails in interesting situations. We have nothing to say anything about unitary, but even the lesser condition that the gauge generators are at all operators typically fails for local symmetries.

Consider for concreteness the case of Yang-Mills theory, where the gauge generators satisfy the current algebra $[J_a(m), J_b(n)] = f_{ab}{}^c J_c(m+n)$, with notation as in (2.12). The algebra typically acts linearly on the Fourier components of matter fields, i.e. as $[J_a(m), \phi(n)] = T_a\phi(m+n)$, where the $T_a$ are representation matrices. If the time component of $m$ is $m_0$, the normal-ordered generators read

$$\begin{aligned} J_a(m) &= \sum_{n\in\mathbb{Z}^N} :\pi(n)T_a\phi(m-n): \\ &= \sum_{n_0<m_0/2} \pi(n)T_a\phi(m-n) + \sum_{n_0>m_0/2} T_a\phi(m-n)\pi(n). \end{aligned} \tag{3.35}$$

They satisfy the algebra

$$[J_a(m), J_b(n)] = f_{ab}{}^c J_c(m+n) + \mathrm{tr}(T_aT_b)\ \delta_{m+n} \sum_{0\leqslant n_0<m_0} 1. \tag{3.36}$$

In one dimension, the last sum is $\sum_{n_0=0}^{m_0-1} 1 = m_0$, and the algebra is recognized as an affine Kac-Moody algebra. In more dimensions, however, the sum diverges. More explicitly, it becomes

$$\sum_{0\leqslant n_0<m_0} 1 = \sum_{n_0=0}^{m_0-1} 1 \cdot \sum_{n_1=-\infty}^{\infty} 1 \cdot ... = m_0 \cdot \infty^{N-1}, \tag{3.37}$$

where $\infty$ in the last line stands for the number of integers. The appearence of an infinite central extension is of course nonsense, and it shows that the normal-ordered $J_a(m)$'s are not operators.

# 4 Covariant quantization

The next step in [12] was to introduce *the observer's trajectory* $q^\mu(t)$, expand all fields in a Taylor series around it, and quantize in the space of

Taylor data histories. The motivation was mainly aesthetic; by adding the observer's trajectory, it is possible to write down a covariant expression for the Hamiltonian, namely as the operator which translates the fields relative to the observer. However, it is in the presence of gauge symmetries that this construction becomes indispensable.

As we saw in the previous section, not only do quantum effects ruin nilpotency of the BRST operator, but they make the gauge generators ill defined. However, it is possible to regularize the theory formulated in terms of Taylor data, in such a way that the full gauge symmetry of the original model is preserved, and the regularized gauge generators are well-defined operators. The price to pay is the appearance of an anomaly.

In the first step, we pass to a parametrized theory, and add a parameter $t$ to each field. In other words, we make the replacements $\phi^\alpha \to \phi^\alpha(t)$, $\phi^*_\alpha \to \phi^*_\alpha(t)$, etc. Contrary to the discussion in the previous section, this $t$ is not included among the original $\alpha$'s. Hence $t$ is an unphysical parameter which must eventually be removed. In order to implement $t$-independence in cohomology, we add the corresponding differential. To keep the notation compact, denote collectively by $\phi^A = (\phi^\alpha, \phi^*_\alpha, \zeta_a, c^a)$ the set of all fields and antifields. We assume that there is a nilpotent differential $\delta$, which may be KT or BRST, such that

$$\delta\phi^A = z^A(\phi). \tag{4.1}$$

Nilpotency leads to the condition

$$\delta^2\phi^A = z^B\partial_B z^A \equiv 0. \tag{4.2}$$

Now replace $\phi^A \to \phi^A(t)$ and introduce addional antifields and antighosts $\bar\phi^A(t)$, whose job is to cancel $t$-dependence in cohomology. Define two differentials $\delta$ and $\sigma$ by

$$\begin{aligned}
\delta\phi^A(t) &= z^A(t), \\
\delta\bar\phi^A(t) &= \bar\phi^B(t)\partial_B z^A(t), \\
\sigma\phi^A(t) &= 0, \\
\sigma\bar\phi^A(t) &= \partial_t\phi^A(t),
\end{aligned} \tag{4.3}$$

where $\partial_t = \partial/\partial t$ is the $t$ derivative. One verifies that $\delta^2\bar\phi^A(t) = 0$, that $\sigma^2 = 0$ and that $\delta\sigma = -\sigma\delta$. Since $\delta^2\phi^A(t) = 0$ by assumption, the combined differential $\delta+\sigma$ is nilpotent. The $\sigma$ cohomology is readily computed. $H^0(\sigma)$ is spanned by functionals $\phi^A(t)$ which satisfy $\partial_t\phi^A(t) = 0$, i.e. $t$-independent

functions. Hence $H^0(\delta+\sigma) = H^0(\delta)$. This result is of course not unexpected. Nothing has neither been gained nor lost by first adding a parameter $t$ and then immediately removing it in cohomology. The reason for this exercise is rather that $t$-dependent fields arise from Taylor expansions, and whereas physical fields must be independent of $t$.

We are really interested in field theories, where the fields and antifield $\phi^A(x)$ depend on the spacetime coordinate $x \in \mathbb{R}^N$. The parametrized fields $\phi^A(x,t)$ can be expanded in a Taylor series around the observer's trajectory $q^\mu(t) \in \mathbb{R}^N$:

$$\phi^A(x,t) = \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} \phi^A_{,\mathbf{m}}(t)(x - q(t))^{\mathbf{m}}, \tag{4.4}$$

where $\mathbf{m} = (m_1, m_2, ..., m_N)$, all $m_\mu \geqslant 0$, is a multi-index of length $|\mathbf{m}| = \sum_{\mu=1}^N m_\mu$, $\mathbf{m}! = m_1! m_2! ... m_N!$, and

$$(x - q(t))^{\mathbf{m}} = (x^1 - q^1(t))^{m_1}(x^2 - q^2(t))^{m_2} ... (x^N - q^N(t))^{m_N}. \tag{4.5}$$

Denote by $\mu$ a unit vector in the $\mu$:th direction, so that $\mathbf{m} + \mu = (m_1, ..., m_\mu + 1, ..., m_N)$. The Taylor coefficient

$$\phi^A_{,\mathbf{m}}(t) = \partial_{\mathbf{m}} \phi^A(q(t),t) = \underbrace{\partial_1 .. \partial_1}_{m_1} .. \underbrace{\partial_N .. \partial_N}_{m_N} \phi^A(q(t),t) \tag{4.6}$$

is recognized as the $|\mathbf{m}|$:th order mixed partial derivative of $\phi^A(x,t)$, evaluated on the observer's trajectory $q^\mu(t)$.

The Taylor coefficients $\phi^A_{,\mathbf{m}}(t)$ are referred to as *jets*; more precisely, infinite jets. Similarly, we define a *p-jet* by truncation to $|\mathbf{m}| \leqslant p$; this will play an important role as a regularization of the symmetry generators. Expand also the Euler-Lagrange equations and the constraints in a similar Taylor series,

$$\begin{aligned} \mathcal{E}_\alpha(x,t) &= \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} \mathcal{E}_{\alpha,\mathbf{m}}(t)(x - q(t))^{\mathbf{m}}, \\ r^\alpha_a(x,t) &= \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} r^\alpha_{a,\mathbf{m}}(t)(x - q(t))^{\mathbf{m}}, \end{aligned} \tag{4.7}$$

etc. These relations define the jets $\mathcal{E}_{\alpha,\mathbf{m}}(t)$ and $r^\alpha_{a,\mathbf{m}}(t)$. Given two jets $f_{,\mathbf{m}}(t)$ and $g_{,\mathbf{m}}(t')$, we define their product

$$(f(t)g(t'))_{,\mathbf{m}} = \sum_{\mathbf{n}} \binom{\mathbf{m}}{\mathbf{n}} f_{,\mathbf{n}}(t) g_{,\mathbf{m}-\mathbf{n}}(t'). \tag{4.8}$$

It is clear that $(f(t)g(t'))_{,\mathbf{m}}$ is the jet corresponding to the field $f(x,t)g(x,t')$. For brevity, we also denote $(fg)_{,\mathbf{m}}(t) = (f(t)g(t))_{,\mathbf{m}}$.

The equations of motion, the constraints, and the time-independence condition translate into

$$\begin{aligned} \mathcal{E}_{\alpha,\mathbf{m}}(t) &= 0, \\ \int dt' \; (r^\alpha_a(t,t')\mathcal{E}_\alpha(t'))_{,\mathbf{m}} &= 0, \\ D_t\phi^\alpha_{,\mathbf{m}}(t) \equiv \dot{\phi}^\alpha_{,\mathbf{m}}(t) - \sum_\mu \dot{q}^\mu(t)\phi^\alpha_{,\mathbf{m}+\mu}(t) &= 0. \end{aligned} \tag{4.9}$$

The BRST differential $s$ which implements these conditions is obtained from (3.25) by Taylor expansion:

$$\begin{aligned} sc^a_{,\mathbf{m}}(t) &= -\frac{1}{2}\iint dt'dt'' \; (f_{bc}{}^a(t,t',t'')c^b(t')c^c(t''))_{,\mathbf{m}}, \\ s\phi^\alpha_{,\mathbf{m}}(t) &= \int dt' \; (r^\alpha_a(t,t')c^a(t'))_{,\mathbf{m}}, \\ s\phi^*_{\alpha,\mathbf{m}}(t) &= \mathcal{E}_{\alpha,\mathbf{m}}(t) + \iint dt'dt''(\partial_\alpha r^\beta_a(t,t',t'')\phi^*_\beta(t')c^a(t''))_{,\mathbf{m}}, \\ s\zeta_{a,\mathbf{m}}(t) &= \int dt' \; (r^\alpha_a(t,t')\phi^*_\alpha(t'))_{,\mathbf{m}} \\ &\quad - \iint dt'dt'' \; (f_{ab}{}^c(t,t',t'')\zeta_c(t')c^b(t''))_{,\mathbf{m}}. \end{aligned} \tag{4.10}$$

The classical cohomology group $H^0_{cl}(s)$ consists of linear combinations of jets $\phi^\alpha_{,\mathbf{m}}(t)$ satisfying the equations (4.9) modulo gauge transformations.

In [12], we also discussed the equation of motion for the observer's trajectory. In Minkowski spacetime, it is natural to assume that $q^\mu(t)$ satisfies the geodesic equation $\ddot{q}^\mu(t) = 0$, which gives a contribution to the BRST differential

$$sq^*_\mu(t) = \eta_{\mu\nu}\ddot{q}^\nu(t). \tag{4.11}$$

$H^0_{cl}(s)$ only contains trajectories which are straight lines,

$$q^\mu(t) = u^\mu t + a^\mu, \tag{4.12}$$

where $u^\mu$ and $a^\mu$ are constant vectors and $u_\mu u^\mu = 1$. This condition fixes the scale of the parameter $t$ in terms of the Minkowski metric, so we may regard it as proper time rather than as an arbitrary parameter.

Now introduce the canonical jet momenta $\pi_A^{,\mathbf{m}}(t)$, and momenta $p_\mu(t)$ and $p_*^\mu(t)$ for the observer's trajectory and its antifield. The non-zero brackets are

$$
\begin{aligned}
[\pi_A^{,\mathbf{m}}(t), \phi^B_{,\mathbf{n}}(t')] &= \delta^B_A \delta^{\mathbf{m}}_{\mathbf{n}} \delta(t-t'), \\
[p_\mu(t), q^\nu(t')] &= \delta^\nu_\mu \delta(t-t'), \\
[p_*^\mu(t), q^*_\nu(t')] &= \delta^\mu_\nu \delta(t-t').
\end{aligned}
\tag{4.13}
$$

As in [12], we now define a genuine Hamiltonian $H$, which translates the fields relative to the observer or vice versa. Since the formulas are shortest when $H$ acts on the trajectory but not on the jets, we make that choice, and define

$$
H = i \int dt\ (\dot{q}^\mu(t) p_\mu(t) + \dot{q}^*_\mu(t) p_*^\mu(t)).
\tag{4.14}
$$

Note the sign; moving the fields forward in $t$ is equivalent to moving the observer backwards. This Hamiltonian acts on the jets as

$$
\begin{aligned}
&[H, q^\mu(t)] = i\dot{q}^\mu(t), \\
&[H, q^*_\mu(t)] = i\dot{q}^*_\mu(t), \\
&[H, \phi^A_{,\mathbf{m}}(t)] = [H, \bar{\phi}^A{}_{,\mathbf{m}}(t)] = 0.
\end{aligned}
\tag{4.15}
$$

Substituting this formula into (4.4), we get the energy of the fields from

$$
[H, \phi^A(x,t)] = -i\dot{q}^\mu(t) \partial_\mu \phi^A(x,t).
\tag{4.16}
$$

In Minkowski space, the trajectory is the straight line (4.12), and $\dot{q}^\mu(t) = u^\mu$. If we take $u^\mu$ to be the constant four-vector $u^\mu = (1,0,0,0)$, then (4.16) reduces to

$$
[H, \phi^A(x,t)] = -i \frac{\partial}{\partial x^0} \phi^A(x,t).
\tag{4.17}
$$

The Hamiltonian (4.14) is thus a truly covariant generalization of the energy operator.

Now we quantize the theory. Since all operators depend on the parameter $t$, we can define the Fourier components, e.g.

$$
\begin{aligned}
\phi^A_{,\mathbf{m}}(t) &= \int_{-\infty}^{\infty} dm\ \phi^A_{,\mathbf{m}}(m) \mathrm{e}^{imt}, \\
q^\mu(t) &= \int_{-\infty}^{\infty} dm\ q^\mu(m) \mathrm{e}^{imt}.
\end{aligned}
\tag{4.18}
$$

The Fock vacuum $\big|0\big\rangle$ is defined to be annihilated by all negative frequency modes, i.e.

$$\begin{aligned}&\phi^A_{,\mathbf{m}}(-m)\big|0\big\rangle = \pi^{,\mathbf{m}}_A(-m)\big|0\big\rangle = 0, \\ &q^\mu(-m)\big|0\big\rangle = q^*_\mu(-m)\big|0\big\rangle = p_\mu(-m)\big|0\big\rangle = p^\mu_*(-m)\big|0\big\rangle = 0,\end{aligned} \tag{4.19}$$

for all $-m < 0$. The normal-ordered form of the Hamiltonian (4.14) reads, in Fourier space,

$$H = -\int_{-\infty}^{\infty} dm\ m(\,{:}q^\mu(m)p_\mu(-m){:}\, +\, {:}q^*_\mu(m)p_\mu(-m){:}\,), \tag{4.20}$$

where double dots indicate normal ordering with respect to frequency. This ensures that $H\big|0\big\rangle = 0$.

Classically, the kinematical phase space is identified with the KT cohomology group $H^0_{cl}(\delta)$, i.e. the the space of fields $\phi^\alpha(x)$ which solve $\mathcal{E}_\alpha(x) = 0$, and trajectories $q^\mu(t) = u^\mu t + a^\mu$, where $u^2 = 1$. In the physical phase space points on gauge orbits are identified, so we identify it with the BRST cohomology group $H^0_{cl}(s)$. After quantization, the fields and trajectories become operators, which act on the history Fock space. If the quantum BRST operator is nilpotent, we identify the physical Hilbert space with the BRST state cohomology $H^0_{qm}(Q_{BRST})$, which is the space of functions of the positive-energy modes of the classical physical phase space variables. However, if $\{Q_{BRST}, Q_{BRST}\} \neq 0$ we have an anomaly, and the BRST cohomology is not well-defined on the quantum level. Instead, we must then identify the physical Hilbert space with the KT cohomology $H^0_{qm}(Q_{KT})$; this is always well-defined because $Q_{KT}$ is always nilpotent.

We can explicitly write down the KT operator (3.33) and longitudinal operator (3.34) in jet space, and thus also $Q_{BRST} = Q_{KT} + Q_{Long}$. If the EL equations $\mathcal{E}_\alpha$ are of order 2 and the Noether identities $r^\alpha_a\mathcal{E}_\alpha$ of order 3,

they read

$$\begin{aligned}
Q_{KT} &= \sum_{|\mathbf{m}|\leqslant p-2} \int dt \ :\mathcal{E}_{\alpha,\mathbf{m}}(t)\pi_*^{\alpha,\mathbf{m}}(t): \qquad (4.21)\\
&+ \sum_{|\mathbf{m}|\leqslant p-3} \int dtdt' \ :(r_a^\alpha(t,t')\phi_\alpha^*(t'))_{,\mathbf{m}}\chi^{a,\mathbf{m}}(t):, \\
Q_{Long} &= \sum_{|\mathbf{m}|\leqslant p-2} \int dtdt' \ :(r_a^\alpha(t,t')c^a(t'))_{,\mathbf{m}}\pi_\alpha^{;\mathbf{m}}(t): \qquad (4.22)\\
&+ \sum_{|\mathbf{m}|\leqslant p-2} \int dtdt'dt'' \ :(\partial_\alpha r_a^\beta(t,t',t'')\phi_\beta^*(t')c^a(t''))_{,\mathbf{m}}\pi_*^{\alpha,\mathbf{m}}(t): \\
&- \sum_{|\mathbf{m}|\leqslant p-2} \int dtdt'dt'' \ :(f_{ab}{}^c(t,t',t'')\zeta_c(t')c^b(t''))_{,\mathbf{m}}\chi^{a,\mathbf{m}}(t): \\
&- \frac{1}{2}\sum_{|\mathbf{m}|\leqslant p-2} \int dtdt'dt'' \ :(f_{ab}{}^c(t,t',t'')c^a(t')c^b(t''))_{,\mathbf{m}}b_c^{;\mathbf{m}}(t):.
\end{aligned}$$

The condition for $Q^2_{Long} = 0$, and thus $Q^2_{BRST} = 0$, is that the algebra generated by the normal-ordered gauge generators is anomaly free. Necessary conditions for this are discussed in the next section.

# 5 Gauge algebra

Let us now discuss the crucial issue how the algebra of gauge transformations is represented at the quantum level. We are dealing with two separate questions: are the gauge generators well-defined operators after quantization, and if so, is $Q^2_{BRST} = 0$? For simplicity, we assume that the constraint functions and the structure functions have the special forms

$$\begin{aligned}
r_a^\alpha(t,t') &= r_a^\alpha(t)\delta(t-t'), \\
f_{ab}{}^c(t,t',t'') &= f_{ab}{}^c\delta(t-t'')\delta(t'-t'').
\end{aligned} \qquad (5.1)$$

The constraints (3.23) and the Lie algebra (3.24) then become

$$\begin{aligned}
r_a^\alpha(t)\mathcal{E}_\alpha(t) &\equiv 0, \\
[J_a(t), J_b(t')] &= f_{ab}{}^c J_c(t)\delta(t-t').
\end{aligned} \qquad (5.2)$$

These assumptions are true for the gauge algebras considered in this paper. In the non-covariant form, $t = x^0$ is the time coordinate, whereas the space

coordinates $x^i$ are included among the indices $a$. In the covariant formulation in jet space, $t$ is the parameter along the observer's trajectory, whereas $a$ contains the multi-indices $\mathbf{m}$ labelling mixed partial derivatives. However, the assumption (5.2) does not hold for the diffeomorphism algebra relevant to gravity.

The gauge generator (3.17) becomes $J_a(t) = J^1_a(t)+J^2_a(t)+J^3_a(t)+J^4_a(t)$, where

$$\begin{aligned} J^1_a(t) &= :r^\alpha_a(t)\pi_\alpha(t): \\ J^2_a(t) &= -\partial_\alpha r^\beta_a(t):\phi^*_\beta(t)\pi^\alpha_*(t):, \\ J^3_a(t) &= f_{ab}{}^c:\zeta_c(t)\chi^b(t):, \\ J^4_a(t) &= -f_{ab}{}^c:c^b(t)b_c(t):. \end{aligned} \tag{5.3}$$

As usual, the double dots denote normal ordering w.r.t. frequency. Let us focus on the first term, which explicitly reads

$$J^1_a(t) = r^\alpha_a(t)\pi^<_\alpha(t) + \pi^>_\alpha(t)r^\alpha_a(t). \tag{5.4}$$

where

$$\begin{aligned} \pi_\alpha(t) &= \int_{-\infty}^{\infty} dm\ \pi_\alpha(m)\mathrm{e}^{imt} \equiv \pi^<_\alpha(t) + \pi^>_\alpha(t), \\ \pi^<_\alpha(t) &= \int_{-\infty}^{0} dm\ \pi_\alpha(m)\mathrm{e}^{imt}, \\ \pi^>_\alpha(t) &= \int_{0}^{\infty} dm\ \pi_\alpha(m)\mathrm{e}^{imt}. \end{aligned} \tag{5.5}$$

We also need to mode-expand the delta-function,

$$\begin{aligned} \delta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dm\ \mathrm{e}^{imt} \equiv \delta^<(t) + \delta^>(t), \\ \delta^<(t) &= \frac{1}{2\pi}\int_{-\infty}^{0} dm\ \mathrm{e}^{imt}, \\ \delta^>(t) &= \frac{1}{2\pi}\int_{0}^{\infty} dm\ \mathrm{e}^{imt}. \end{aligned} \tag{5.6}$$

Now we further assume that

$$[\pi_\alpha(t), r^\beta_b(t')] = \partial_\alpha r^\beta_b(t')\delta(t-t'), \tag{5.7}$$

from which it follows that

$$[\pi^<_\alpha(t), r^\beta_b(t')] = \partial_\alpha r^\beta_b(t')\delta^<(t-t'). \tag{5.8}$$

The assumption (5.7) does hold in the gauge algebra examples below. Using the identity

$$\delta^{>}(t-t')\delta^{<}(t'-t) - \delta^{>}(t'-t)\delta^{<}(t-t') = \frac{1}{2\pi i}\dot{\delta}(t-t'), \tag{5.9}$$

it is straightforward to show that the $J_a^1$'s satisfy an extension of the gauge algebra,

$$[J_a^1(t), J_b^1(t')] = f_{ab}{}^c J_c^1(t)\delta(t-t') + \frac{1}{2\pi i}\partial_\alpha r_b^\beta(t')\partial_\beta r_a^\alpha(t)\dot{\delta}(t-t'). \tag{5.10}$$

This is a well-defined expression provided that there are only finitely many degrees of freedom for each $t$, so the sums over $\alpha$ and $\beta$ are finite. This is the situation encountered in quantum-mechanical systems, and also in field theories in one dimension. For field theories in several dimensions, this condition is not satisfied. Even leaving the question of anomaly freedom aside, we need to make the extension finite in order for $J_a^1$ to be a well-defined operator. The way to do this is to truncate the Taylor expansion (4.4) at order $p$. In this way, we replace the field $\phi^\alpha(x,t)$, which has infinitely many components for each $t$ (labelled by $x \in \mathbb{R}^N$), by the $p$-jet $\phi^\alpha_{,\mathbf{m}}(t)$, which has finitely many components for each $t$ (the number of multi-indices with $|\mathbf{m}| \leqslant p$ equals $\binom{N+p}{N}$).

If we write the extended algebra as

$$[J_a(t), J_b(t')] = f_{ab}{}^c J_c(t-t') + \text{ext }(J_a(t), J_b(t')), \tag{5.11}$$

and we observe that fermions give opposite signs, we find that the extensions are

$$\begin{aligned}
\text{ext }(J_a^1(t), J_b^1(t')) &= \frac{1}{2\pi i}\partial_\alpha r_b^\beta(t')\partial_\beta r_a^\alpha(t)\dot{\delta}(t-t'), \\
\text{ext }(J_a^2(t), J_b^2(t')) &= -\frac{1}{2\pi i}\partial_\alpha r_b^\beta(t'))\partial_\beta r_a^\alpha(t)\dot{\delta}(t-t'), \\
\text{ext }(J_a^3(t), J_b^3(t')) &= \frac{1}{2\pi i}f_{ac}{}^d f_{bd}{}^c\dot{\delta}(t-t'), \\
\text{ext }(J_a^4(t), J_b^4(t')) &= -\frac{1}{2\pi i}f_{ac}{}^d f_{bd}{}^c\dot{\delta}(t-t').
\end{aligned} \tag{5.12}$$

Naïvely, ext $(J_a^1(t), J_b^1(t')) = -$ext $(J_a^2(t), J_b^2(t'))$ and ext $(J_a^3(t), J_b^3(t')) = -$ext $(J_a^4(t), J_b^4(t'))$, so it would seem that the total extension cancels. However, a more careful treatment in jet space shows that this is not the case. If $\phi^\alpha(x)$ is a bosonic field, the corresponding jet $\phi^\alpha_{,\mathbf{m}}(t)$ is defined for $|\mathbf{m}| \leqslant p$.

The EL equations are second order and the Noether identities $r_a^\alpha \mathcal{E}_\alpha$ are third order, so the antifield $\phi^*_{\alpha,\mathbf{m}}$ is defined for $|\mathbf{m}| \leqslant p-2$, and the second-order antifield $\zeta_{a,\mathbf{m}}$ for $|\mathbf{m}| \leqslant p-3$. Finally, the ghost $c^a(x)$ must typically be defined for $|\mathbf{m}| \leqslant p+1$. Hence (5.3) should be written more carefully as

$$
\begin{aligned}
J_a^1(t) &= \sum_{|\mathbf{m}|\leqslant p} {:} r^\alpha_{a,\mathbf{m}}(t)\pi_\alpha^{;\mathbf{m}}(t){:} \\
J_a^2(t) &= -\sum_{|\mathbf{m}|\leqslant p-2} {:} (\partial_\alpha r_a^\beta \phi^*_\beta)_{,\mathbf{m}}(t)\pi_*^{\alpha,\mathbf{m}}(t){:}\,, \\
J_a^3(t) &= f_{ab}{}^c \sum_{|\mathbf{m}|\leqslant p-3} {:} \zeta_{c,\mathbf{m}}(t)\chi^{b,\mathbf{m}}(t){:}\,, \\
J_a^4(t) &= -f_{ab}{}^c \sum_{|\mathbf{m}|\leqslant p+1} {:} c^b_{,\mathbf{m}}(t) b_c^{;\mathbf{m}}(t){:}\,.
\end{aligned}
\tag{5.13}
$$

In analogy with (5.7), we assume that

$$
[\pi_\alpha^{;\mathbf{m}}(t), r^\beta_{b,\mathbf{n}}(t')] = \partial_\alpha^{\mathbf{m}} r^\beta_{b,\mathbf{n}}(t')\delta(t-t'). \tag{5.14}
$$

The jet form of the extensions (5.12) read

$$
\begin{aligned}
\mathrm{ext}\ (J_a^1(t), J_b^1(t')) &= \frac{1}{2\pi i} \sum_{|\mathbf{m}|\leqslant p} \sum_{|\mathbf{n}|\leqslant p} \partial_\alpha^{\mathbf{m}} r^\beta_{b,\mathbf{n}}(t')\partial_\beta^{\mathbf{n}} r^\alpha_{a,\mathbf{m}}(t)\dot\delta(t-t'), \\
\mathrm{ext}\ (J_a^2(t), J_b^2(t')) &= -\frac{1}{2\pi i} \sum_{|\mathbf{m}|\leqslant p-2} \sum_{|\mathbf{n}|\leqslant p-2} \partial_\alpha^{\mathbf{m}} r^\beta_{b,\mathbf{n}}(t')\partial_\beta^{\mathbf{n}} r^\alpha_{a,\mathbf{m}}(t)\dot\delta(t-t'), \\
\mathrm{ext}\ (J_a^3(t), J_b^3(t')) &= \frac{1}{2\pi i} f_{ac}{}^d f_{bd}{}^c \binom{N+p-3}{N} \dot\delta(t-t'), \\
\mathrm{ext}\ (J_a^4(t), J_b^4(t')) &= -\frac{1}{2\pi i} f_{ac}{}^d f_{bd}{}^c \binom{N+p+1}{N} \dot\delta(t-t').
\end{aligned}
\tag{5.15}
$$

Note in particular that $f_{ac}{}^d f_{bd}{}^c$ is proportional to the second Casimir operator.

The appearance of gauge anomalies is thus a generic feature of canonical quantization of field theories in history space. In exceptional cases, such as the free Maxwell field to be discussed next, there are no anomalies, and we can pass to the BRST cohomology, but generically there are anomalies and we must identify the physical Hilbert space with the KT cohomology only. Moreover, the crucial nature of the truncation to $p$-jets becomes clear;

in (5.12) the sums over $\alpha$ and $\beta$ diverge (for field theories), whereas the sums over $\mathbf{m}$, $\mathbf{n}$, $\alpha$ and $\beta$ in (5.15) are finite and thus well defined. The sums diverge in the $p \to \infty$ limit, however, making it difficult to remove the regularization.

# 6 The free Maxwell field

The Maxwell field $A_\mu(x)$ transforms as a vector field under the Poincaré group and as a connection under the gauge algebra $\mathfrak{map}(N, \mathfrak{u}(1))$, whose smeared generators are denoted by $\mathcal{J}_X = \int d^N x \; X(x) J(x)$:

$$[\mathcal{J}_X, A_\mu(x)] = \partial_\mu X(x). \tag{6.1}$$

We use the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ to freely raise and lower indices, e.g. $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}$. The field strength $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ transforms in the adjoint representation, i.e. trivially. The action

$$S = \frac{1}{4}\int d^N x \; F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{6.2}$$

leads to the equations of motion

$$\mathcal{E}^\mu(x) \equiv -\frac{\delta S}{\delta A_\mu(x)} = \partial_\nu F^{\mu\nu}(x) = 0. \tag{6.3}$$

The Maxwell equations are not all independent, because of the constraints

$$\partial_\mu \mathcal{E}^\mu(x) = \partial_\mu \partial_\nu F^{\mu\nu}(x) \equiv 0. \tag{6.4}$$

We are thus instructed to introduce the following fields: the first-order antifield $A^\mu_*(x)$ for the EL equation $\partial_\nu F^{\mu\nu}(x) = 0$; the second-order antifield $\zeta(x)$ for the identity $\partial_\mu\partial_\nu F^{\mu\nu}(x) \equiv 0$; and the ghost $c(x)$ to identify fields related by a gauge transformation of the form (6.1).

The BRST operator $s$ acts as

$$\begin{aligned} sc(x) &= 0, \\ sA_\mu(x) &= \partial_\mu c(x), \\ sA^\mu_*(x) &= \partial_\nu F^{\mu\nu}(x), \\ s\zeta(x) &= \partial_\mu A^\mu_*(x), \end{aligned} \tag{6.5}$$

We check that $sF_{\mu\nu} = s\partial_\mu A^\mu_* = 0$, so the kernel of $s$ is spanned by $c$, the field strengths $F_{\mu\nu}$, and $\partial_\mu A^\mu_*$. im $s$ is generated by the ideals $\partial_\mu c$, $\partial_\nu F^{\mu\nu}$,

and $\partial_\mu A^\mu_*$. Hence $H^\bullet_{cl}(s)$ consists of the gauge-invariant parts of $A_\mu$ (i.e. $F_{\mu\nu}$) which solve the Maxwell equations, as expected.

Introduce canonical momenta $E^\mu(x)$, $E^*_\mu(x)$, $\chi(x)$ and $b(x)$, defined by the following non-zero brackets:

$$\begin{array}{rcl} [E^\mu(x), A_\nu(x')] & = & \delta^\mu_\nu \delta(x-x'), \\ \{E^*_\mu(x), A^\nu_*(x')\} & = & \delta^\nu_\mu \delta(x-x'), \\ [\chi(x), \zeta(x')] & = & \delta(x-x'), \\ \{b(x), c(x')\} & = & \delta(x-x'). \end{array} \tag{6.6}$$

It should be emphasized that $E^\mu = \delta/\delta A_\mu$ is the conjugate of the gauge potential in history space, and not yet related to the electric field $F^{\mu 0}$. We could introduce the condition $E^\mu = F^{\mu 0}$ as a constraint in the history phase space, turning the Maxwell equations into second class constraints. By keeping dynamics as a first-class constraint no such condition, which would ruin covariance, is necessary. The BRST operator can explicitly be written as

$$Q_{BRST} = \int d^N x \, \Big( \partial_\mu c(x) E^\mu(x) + \partial_\nu F^{\mu\nu}(x) E^*_\mu(x) + \partial_\mu A^\mu_*(x)\chi(x) \Big). \tag{6.7}$$

From this we read off that the action on the momenta is given by

$$\begin{array}{rcl} sb(x) & = & \partial_\mu E^\mu(x), \\ sE^\mu(x) & = & \partial_\mu \partial^\nu E^*_\nu(x) - \partial_\nu \partial^\nu E^*_\mu(x), \\ sE^*_\mu(x) & = & \partial_\mu \chi(x), \\ s\chi(x) & = & 0. \end{array} \tag{6.8}$$

Note the duality between fields and momenta; (6.5) and (6.8) are identified under the replacements

$$\begin{array}{rcl} b & \longleftrightarrow & \zeta, \\ E^\mu & \longleftrightarrow & A^\mu_*, \\ E^*_\mu & \longleftrightarrow & A_\mu, \\ \chi & \longleftrightarrow & c. \end{array} \tag{6.9}$$

The physical content of the theory is clearer in Fourier space. The BRST operator

$$\begin{array}{rcl} Q_{BRST} & = & \displaystyle\int d^N k \, \Big( k_\mu c(k) E^\mu(-k) + (k^\mu k_\nu A^\nu(k) - k^\nu k_\nu A^\mu(k)) E^*_\mu(-k) \\ & & \qquad + k_\mu A^\mu_*(k) \chi(-k) \Big), \end{array} \tag{6.10}$$

acts on the Fourier modes as

$$
\begin{aligned}
sc(k) &= 0, \\
sA_\mu(k) &= k_\mu c(k), \\
sA^\mu_*(k) &= k^\mu k_\nu A^\nu(k) - k^\nu k_\nu A^\mu(k), \\
s\zeta(k) &= k_\mu A^\mu_*(k).
\end{aligned}
\tag{6.11}
$$

We distinguish between two cases:

1. $k^2 = \omega^2 \neq 0$, say $k = (\omega, 0, 0, 0)$. Then $sc = 0$, $sA_0 = \omega c$, $sA_1 = sA_2 = sA_3 = 0$, $sA^0_* = \omega^2 A_0 - \omega\omega A_0 = 0$, $sA^1_* = \omega^2 A_1$ $sA^2_* = \omega^2 A_2$, $sA^3_* = \omega^2 A_3$ and $s\zeta = \omega A^0_*$. The kernel is thus spanned by $c$, $A_1$, $A_2$, $A_3$ and $A^0_*$, and the image is spanned by $c$, $A_1$, $A_2$, $A_3$ and $A^0_*$. Since $\ker s = \mathrm{im}\ s$ there is no cohomology.

2. $k^2 = 0$, say $k = (k_0, 0, 0, k_0)$. Then $sc = 0$, $sA_0 = sA_3 = k_0 c$, $sA_1 = sA_2 = 0$, $sA^0_* = sA^3_* = k^0 k_\nu A^\nu$, $sA^1_* = sA^2_* = 0$ and $s\zeta = k_\mu A^\mu_*$. The kernel is thus spanned by $c$, $A_1$, $A_2$, $k^\mu A_\mu = A_0 - A_3$, $A^1_*$, $A^2_*$, and $k_\mu A^\mu_* = A^0_* - A^3_*$. The image is spanned by $c$, $k^\mu A_\mu$ and $k_\mu A^\mu_*$, which factor out in cohomology. We are left with two physical polarizations $A_1$ and $A_2$.

We now quantize in the history phase space before introducing dynamics by passing to the BRST cohomology. We single out one direction $x^0$ as time, and take the Hamiltonian to be the generator of rigid time translations,

$$
\begin{aligned}
H &= -i \int d^N x\ \Big(\partial_0 A_\mu(x) E^\mu(x) + \partial_0 A^\mu_*(x) E^*_\mu(x) \\
&\qquad + \partial_0 \zeta(x)\chi(x) + \partial_0 c(x) b(x)\Big) \\
&= \int d^N k\ k_0 \Big(A_\mu(k) E^\mu(-k) + A^\mu_*(k) E^*_\mu(-k) \\
&\qquad + \zeta(k)\chi(-k) + c(k) b(-k)\Big).
\end{aligned}
\tag{6.12}
$$

Note that at this stage we break Poincaré invariance, since the Hamiltonian treats the $x^0$ coordinate differently from the other $x^\mu$. Quantize by introducing a Fock vacuum $\big|0\big\rangle$ satisfying

$$
\begin{aligned}
&A_\mu(k)\big|0\big\rangle = E^\mu(k)\big|0\big\rangle = A^\mu_*(k)\big|0\big\rangle = E^*_\mu(k)\big|0\big\rangle = \\
&\zeta(k)\big|0\big\rangle = \chi(k)\big|0\big\rangle = c(k)\big|0\big\rangle = b(k)\big|0\big\rangle = 0,
\end{aligned}
\tag{6.13}
$$

for all $k$ such that $k_0 < 0$.

At this point we want to pass to BRST cohomology. There might be problems with normal ordering, but in fact the BRST operator (6.10) is

already normal ordered. This is because the generator of $\mathfrak{u}(1)$ gauge transformations

$$\mathcal{J}_X = -\int d^N x\ X(x)\partial_\mu E^\mu(x) \tag{6.14}$$

is itself already normal ordered. There are thus no anomalies, and the BRST operator (6.10) remains nilpotent. We define the BRST state cohomology as the space of physical states, where a state is physical if it is BRST closed, $Q_{BRST}\big|phys\big\rangle = 0$, and two physical states are equivalent if they differ by a BRST exact state, $\big|phys\big\rangle \sim \big|phys'\big\rangle$ if $\quad \big|phys\big\rangle - \big|phys'\big\rangle = Q_{BRST}\big|\big\rangle$.

The rest proceeds as for the harmonic oscillator [12]. After adding a small perturbation to make the Hessian invertible, all momenta (6.8) vanish in cohomology, and only the transverse polarizations $\epsilon^\mu A_\mu(k) = 0$ with $\epsilon^\mu k_\mu = 0$ and $\epsilon^0 = 0$ survive. A basis for the history Hilbert space consists of multi-quanta states

$$\epsilon_1^\mu A_\mu(k^{(1)}) \ldots \epsilon_n^\mu A_\mu(k^{(n)})\big|0\big\rangle \tag{6.15}$$

where $k_\mu^{(j)} k^{(j)\mu} = 0$ and $k_0^{(j)} > 0$. The energy is given by $H = k_0^{(1)} + \ldots + k_0^{(n)}$. The gauge generators (6.14) act in a well-defined manner, in fact trivially, on the Hilbert space, because $\epsilon_j^\mu k_\mu^{(j)} = 0$.

As in Section 4, we want to give a completely covariant description of the Hamiltonian. To this end, we introduce the observer's trajectory $q^\mu(t) \in \mathbb{R}^N$, and expand all fields in a Taylor series around it, i.e. we pass to jet data. Hence e.g.,

$$A_\mu(x) = \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} A_{\mu,\mathbf{m}}(t)(x - q(t))^{\mathbf{m}}. \tag{6.16}$$

The equations of motion (6.3) translate into

$$\sum_\nu F^{\mu\nu}_{,\mathbf{m}+\nu}(t) = 0, \tag{6.17}$$

and the constraint (6.4) becomes

$$\sum_\mu \mathcal{E}^\mu_{\mu,\mathbf{m}}(t) = \sum_{\mu\nu} F^{\mu\nu}_{,\mathbf{m}+\mu+\nu}(t) \equiv 0, \tag{6.18}$$

where the field strength is

$$F_{\mu\nu,\mathbf{m}}(t) = A_{\mu,\mathbf{m}+\nu}(t) - A_{\nu,\mathbf{m}+\mu}(t) = 0. \tag{6.19}$$

We introduce jets also for the antifields and for the ghost, denoted by $A^\mu_{*,\mathbf{m}}(t)$, $\zeta_{,\mathbf{m}}(t)$, $c_{,\mathbf{m}}(t)$. The BRST differential $s$ which implements all these conditions is defined by

$$\begin{aligned} sc_{,\mathbf{m}}(t) &= 0, \\ sA_{\mu,\mathbf{m}}(t) &= c_{,\mathbf{m}+\mu}(t), \\ sA^\mu_{*,\mathbf{m}}(t) &= \sum_\nu F^{\mu\nu}_{,\mathbf{m}+\nu}(t), \\ s\zeta_{,\mathbf{m}}(t) &= \sum_\mu A^\mu_{*,\mathbf{m}+\mu}(t). \end{aligned} \tag{6.20}$$

Moreover, we demand that the Taylor series does not depend on the parameter $t$, which gives rise to conditions of the type

$$D_t A_{\mu,\mathbf{m}}(t) \equiv \dot{A}_{\mu,\mathbf{m}}(t) - \sum_\nu \dot{q}^\nu(t) A_{\mu,\mathbf{m}+\nu}(t) = 0. \tag{6.21}$$

As explained in (4.3), we need to double the number of antifields and introduce an additional differential $\sigma$ to remove these conditions in cohomology. Thus we introduce antifields $\bar{c}_{,\mathbf{m}}(t)$, $\bar{A}_{\mu,\mathbf{m}}(t)$, $\bar{A}^\mu_{*,\mathbf{m}}(t)$, $\bar{\zeta}_{,\mathbf{m}}(t)$ and set

$$\begin{aligned} \sigma\bar{c}_{,\mathbf{m}}(t) &= \dot{c}_{,\mathbf{m}}(t), \\ \sigma\bar{A}_{\mu,\mathbf{m}}(t) &= \dot{A}_{\mu,\mathbf{m}}(t), \\ \sigma\bar{A}^\mu_{*,\mathbf{m}}(t) &= \dot{A}^\mu_{*,\mathbf{m}}(t), \\ \sigma\bar{\zeta}_{,\mathbf{m}}(t) &= \dot{\zeta}_{,\mathbf{m}}(t) \\ \sigma c_{,\mathbf{m}}(t) &= \sigma A_{\mu,\mathbf{m}}(t) = \sigma A^\mu_{*,\mathbf{m}}(t) = \sigma\zeta_{,\mathbf{m}}(t) = 0. \end{aligned} \tag{6.22}$$

Clearly, $\sigma^2 = 0$. We also extend the definition of the BRST differential $s$ to the barred antifields:

$$\begin{aligned} s\bar{c}_{,\mathbf{m}}(t) &= 0, \\ s\bar{A}_{\mu,\mathbf{m}}(t) &= -\bar{c}_{,\mathbf{m}+\mu}(t), \\ s\bar{A}^\mu_{*,\mathbf{m}}(t) &= -\sum_\nu (\bar{A}_{\nu,\mathbf{m}+\mu}(t) - \bar{A}_{\mu,\mathbf{m}+\nu}(t)), \\ s\bar{\zeta}_{,\mathbf{m}}(t) &= -\sum_\mu \bar{A}^\mu_{*,\mathbf{m}+\mu}(t). \end{aligned} \tag{6.23}$$

That $s^2 = 0$ follows in the same way as for (6.20). Moreover, we verify that $s\sigma = -\sigma s$, and hence $s + \sigma$ is nilpotent.

The classical cohomology group $H^0_{cl}(s+\sigma)$ consists of linear combinations of jets satisfying

$$A_{\mu,\mathbf{m}}(t) = \epsilon_\mu(t)\mathrm{e}^{ik\cdot q(t)}(ik)^{\mathbf{m}} \tag{6.24}$$

where $k^2 = 0$ and the polarization vector $\epsilon_\mu(t)$ is perpendicular both to the photon momentum and the observer's trajectory:

$$\epsilon_\mu(t)k^\mu = \epsilon_\mu(t)\dot{q}^\mu(t) = 0. \tag{6.25}$$

The latter is evidently equivalent to the non-covariant condition $\epsilon^0 = 0$. Moreover, $k \cdot q = k_\mu q^\mu$ and the power $k^{\mathbf{m}}$ is defined in analogy with (4.5). The Taylor series (6.16) can be summed, giving

$$\begin{aligned} A_\mu(x) &= \mathrm{e}^{ik\cdot q(t)} \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!}\epsilon_\mu(t)(ik)^{\mathbf{m}}(x-q(t))^{\mathbf{m}} \\ &= \epsilon_\mu(t)\mathrm{e}^{ik\cdot q(t)}\mathrm{e}^{ik\cdot(x-q(t))} \\ &= \epsilon_\mu(t)\mathrm{e}^{ik\cdot x}. \end{aligned} \tag{6.26}$$

The physical Hamiltonian $H$ is defined as in Equation (4.20). The classical phase space $H^0_{cl}(s+\sigma)$ is thus the the space of plane waves $\mathrm{e}^{ik\cdot x}$, cf. (6.26), and straight trajectories $q^\mu(t) = u^\mu t + a^\mu$. The energy is given by

$$\begin{aligned} [H, \mathrm{e}^{ik\cdot x}] &= k_\mu \dot{q}^\mu(t)\mathrm{e}^{ik\cdot x} = k_\mu u^\mu \mathrm{e}^{ik\cdot x}, \\ [H, q^\mu(t)] &= i\dot{q}^\mu(t) = iu^\mu. \end{aligned} \tag{6.27}$$

This is a covariant description of phase space, because the energy $k_\mu u^\mu$ is Poincaré invariant.

We now quantize the theory before imposing dynamics. To this end, we introduce the canonical momenta for all jets and antijets, and $p_\mu(t)$ and $p^\mu_*(t)$ for the observer's trajectory and its antifield. The defining relations are

$$\begin{aligned} [E^{\mu,\mathbf{m}}(t), A_{\nu,\mathbf{n}}(t')] &= \delta^\mu_\nu \delta^{\mathbf{m}}_{\mathbf{n}} \delta(t-t'), \\ \{E^{*,\mathbf{m}}_\mu(t), A^\nu_{*,\mathbf{n}}(t')\} &= \delta^\nu_\mu \delta^{\mathbf{m}}_{\mathbf{n}} \delta(t-t'), \\ [\chi^{,\mathbf{m}}(t), \zeta_{,\mathbf{n}}(t')] &= \delta^{\mathbf{m}}_{\mathbf{n}} \delta(t-t'), \\ \{b^{,\mathbf{m}}(t), c_{,\mathbf{n}}(t')\} &= \delta^{\mathbf{m}}_{\mathbf{n}} \delta(t-t'), \\ [p_\nu(t), q^\mu(t')] &= \delta^\mu_\nu \delta(t-t'). \end{aligned} \tag{6.28}$$

Since the jets also depend on the parameter $t$, we can define their Fourier components, e.g.

$$\begin{aligned} A_{\mu,\mathbf{m}}(t) &= \int_{-\infty}^{\infty} dm\ A_{\mu,\mathbf{m}}(m)\mathrm{e}^{imt}, \\ q^{\mu}(t) &= \int_{-\infty}^{\infty} dm\ q^{\mu}(m)\mathrm{e}^{imt}. \end{aligned} \tag{6.29}$$

The Fock vacuum (6.13) is replaced by a new vacuum, also denoted by $\big|0\big\rangle$, which is defined to be annihilated by the negative frequency modes of the jets and antijets, e.g.

$$\begin{aligned} &A_{\mu,\mathbf{m}}(-m)\big|0\big\rangle = E^{\mu}_{,\mathbf{m}}(-m)\big|0\big\rangle = A^{\mu}_{*,\mathbf{m}}(-m)\big|0\big\rangle \\ &= E^{*,\mathbf{m}}_{\mu}(-m)\big|0\big\rangle = q^{\mu}(-m)\big|0\big\rangle = p_{\mu}(-m)\big|0\big\rangle = 0, \end{aligned} \tag{6.30}$$

for all $-m < 0$. The quantum Hamiltonian is still defined by (4.20), where double dots indicate normal ordering with respect to frequency, ensuring that $H\big|0\big\rangle = 0$.

It remains to check that the algebra of $\mathfrak{u}(1)$ gauge transformations acts in a well-defined manner before we can pass to the BRST cohomology. Since a gauge potential transforms as

$$[\mathcal{J}_X, A_{\mu,\mathbf{m}}(t)] = \partial_{\mathbf{m}+\mu}X(q(t)) \tag{6.31}$$

we have

$$\mathcal{J}_X = \sum_{\mathbf{m}}\sum_{\mu}\int dt\ \partial_{\mathbf{m}+\mu}X(q(t))E^{\mu,\mathbf{m}}(t). \tag{6.32}$$

There are no contributions from the antifields, since $A^{\mu}_{*}$, $\zeta$ and $c$ all transform trivially under $\mathfrak{map}(N, \mathfrak{u}(1))$. The expression (6.32) is evidently normal ordered as it stands, and consequently there are no gauge anomalies.

The rest proceeds as for the scalar field [12]. We can consider the one-quantum state with momentum $k$ over the true Fock vacuum, $\big|k\big\rangle = \exp(ik\cdot x)\big|0\big\rangle$. This state is not an energy eigenstate, because the Hamiltonian excites a quantum of the observer's trajectory: $H\big|k\big\rangle = k_\mu u^\mu\big|k\big\rangle$. In some approximations, we may treat the observer's trajectory as a classical variable and introduce the macroscopic reference state $\big|0; u, a\big\rangle$, on which $q^\mu(t)\big|0; u, a\big\rangle = (u^\mu t + a^\mu)\big|0; u, a\big\rangle$, where $u^\mu$ and $a^\mu$ are c-numbers rather than quantum operators here. We can then consider a state $\big|k, \epsilon; u, a\big\rangle = \epsilon_\mu(t)\exp(ik\cdot x)\big|0; u, a\big\rangle$ with one quantum over the reference state. The

Hamiltonian gives $H\big|k,\epsilon;u,a\big\rangle = k_\mu u^\mu \big|k,\epsilon;u,a\big\rangle$. In particular, if $u^\mu = (1,0,0,0)$, then the eigenvalue of the Hamiltonian is $k_\mu u^\mu = k_0$, as expected. Moreover, the lowest-energy condition (6.30) ensures that only quanta with positive energy will be excited; if $k_\mu u^\mu < 0$ then $\big|k,\epsilon;u,a\big\rangle = 0$.

# 7 Fermions

In the previous section we described the free Maxwell field in covariant canonical quantization. Apart from the explicit introduction of the observer, which occurred already for the free scalar field, the results were completely standard. In this section we couple the Maxwell field to a Dirac spinor, and encounter new phenomena: the gauge algebra acquires an extension, the BRST operator ceases to be nilpotent, and only the KT operator is well defined.

We use standard notation; $\psi$ is a Dirac spinor and $\bar\psi(x) = \psi^\dagger(x)\gamma^0$ is the Dirac conjugate. The Dirac action is

$$S_D = \int d^Nx\ \bar\psi(x)(\gamma^\mu(i\partial_\mu - eA_\mu(x)) - M)\psi(x), \tag{7.1}$$

where $e$ is the charge of the positron, $M$ its mass, and $A_\mu(x)$ is the gauge potential. The Dirac equation, i.e. the EL equation for $\psi$, reads

$$\begin{aligned} \gamma^\mu(i\partial_\mu - eA_\mu(x))\psi(x) &= M\psi(x), \\ (i\partial_\mu + eA_\mu(x))\bar\psi(x)\gamma^\mu &= -M\bar\psi(x). \end{aligned} \tag{7.2}$$

The free Maxwell equations (6.3) are changed into

$$\mathcal{E}^\mu(x) = \partial_\nu F^{\mu\nu}(x) - j^\mu(x) = 0, \tag{7.3}$$

where the current is

$$j^\mu(x) = e\bar\psi(x)\gamma^\mu\psi(x). \tag{7.4}$$

The EL equations are not all independent, because of the continuity equation

$$\partial_\mu\mathcal{E}^\mu(x) = -\partial_\mu j^\mu(x) \equiv 0. \tag{7.5}$$

To implement the Dirac equation in cohomology, we treat $\psi(x)$ and $\bar\psi(x)$ as independent fields, and introduce the corresponding antifields $\psi^*(x)$ and

$\bar{\psi}^*(x)$. The KT differential acts on the fermions as

$$\begin{aligned}
\delta\psi(x) &= 0, \\
\delta\bar{\psi}(x) &= 0, \\
\delta\psi^*(x) &= \gamma^\mu(i\partial_\mu - eA_\mu(x))\psi(x) - M\psi(x), \\
\delta\bar{\psi}^*(x) &= (i\partial_\mu + eA_\mu(x))\bar{\psi}(x)\gamma^\mu + M\bar{\psi}(x).
\end{aligned} \tag{7.6}$$

Moreover, we also need to add an additional antifield to implement the condition $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, but we leave this implicit for brevity of notation. The presence of the fermions changes how the KT differential acts on the Maxwell field (6.5),

$$\begin{aligned}
\delta A_\mu(x) &= 0, \\
\delta A^\mu_*(x) &= \partial_\nu F^{\mu\nu}(x) - j^\mu(x), \\
\delta\zeta(x) &= \partial_\mu A^\mu_*(x) - ie\bar{\psi}^*(x)\psi(x) - ie\bar{\psi}(x)\psi^*(x).
\end{aligned} \tag{7.7}$$

We check that

$$\delta^2\zeta \;=\; \partial_\mu\partial_\nu F^{\mu\nu} - \partial_\mu j^\mu + e\partial_\mu\bar{\psi}\gamma^\mu\phi + e\bar{\psi}\gamma^\mu\partial_\mu\psi \equiv 0. \tag{7.8}$$

The fermions transform under gauge transformations as

$$\begin{aligned}
[\mathcal{J}_X, \psi(x)] &= -eX(x)\psi(x), \\
[\mathcal{J}_X, \bar{\psi}(x)] &= eX(x)\bar{\psi}(x), \\
[\mathcal{J}_X, \psi^*(x)] &= -eX(x)\psi^*(x), \\
[\mathcal{J}_X, \bar{\psi}^*(x)] &= eX(x)\bar{\psi}^*(x).
\end{aligned} \tag{7.9}$$

The BRST differential thus gives

$$\begin{aligned}
s\psi(x) &= -eX(x)\psi(x)c(x), \\
s\bar{\psi}(x) &= eX(x)\bar{\psi}(x)c(x), \\
s\psi^*(x) &= \gamma^\mu(i\partial_\mu - eA_\mu(x))\psi(x) - M\psi(x) - eX(x)\psi^*(x)c(x), \\
s\bar{\psi}^*(x) &= (i\partial_\mu + eA_\mu(x))\bar{\psi}^*(x)\gamma^\mu + M\bar{\psi}(x) + eX(x)\bar{\psi}(x)c(x).
\end{aligned} \tag{7.10}$$

We introduce canonical momenta $\pi(x) = \delta/\delta\psi(x)$, $\bar{\pi}(x) = \delta/\delta\bar{\psi}(x)$, $\pi_*(x) = \delta/\delta\psi^*(x)$, and $\bar{\pi}_*(x) = \delta/\delta\bar{\psi}^*(x)$, with non-zero brackets

$$\begin{aligned}
&\{\pi(x), \psi(y)\} = \delta(x-y), \qquad && [\pi_*(x), \psi^*(y)] = \delta(x-y), \\
&\{\bar{\pi}(x), \bar{\psi}(y)\} = \delta(x-y), \qquad && [\bar{\pi}_*(x), \bar{\psi}^*(y)] = \delta(x-y).
\end{aligned} \tag{7.11}$$

We can then explicitly write down the KT operator

$$
\begin{aligned}
Q_{KT} &= \int d^N x \, \Big( (\partial_\nu F^{\mu\nu}(x) - j^\mu(x)) E^*_\mu(x) \\
&+ (\partial_\mu A^\mu_*(x) - ie\bar\psi^*(x)\psi(x) - ie\bar\psi(x)\psi^*(x))\chi(x) \\
&+ (\gamma^\mu(i\partial_\mu - eA_\mu(x))\psi(x) - M\psi(x))\pi_*(x) \\
&+ ((i\partial_\mu + eA_\mu(x))\bar\psi(x)\gamma^\mu + M\bar\psi(x))\bar\pi_*(x)\Big),
\end{aligned}
\tag{7.12}
$$

the gauge generators,

$$
\begin{aligned}
\mathcal{J}_X &= \int d^N x \, \Big( \partial_\mu X(x) E(x) - eX(x)\psi(x)\pi(x) + eX(x)\bar\psi(x)\bar\pi(x) \\
&- eX(x)\psi^*(x)\pi_*(x) + eX(x)\bar\psi^*(x)\bar\pi_*(x)\Big),
\end{aligned}
\tag{7.13}
$$

and the BRST operator

$$
\begin{aligned}
Q_{BRST} &= \int d^N x \, \Big( \partial_\mu c(x) E^\mu(x) + (\partial_\nu F^{\mu\nu}(x) - j^\mu(x)) E^*_\mu(x) \\
&+ (\partial_\mu A^\mu_*(x) - ie\bar\psi^*(x)\psi(x) - ie\bar\psi(x)\psi^*(x))\chi(x) \\
&- eX(x)\psi(x)c(x)\pi(x) + eX(x)\bar\psi(x)c(x)\bar\pi(x) \\
&+ (\gamma^\mu(i\partial_\mu - eA_\mu(x))\psi(x) - M\psi(x))\pi_*(x) \\
&+ ((i\partial_\mu + eA_\mu(x))\bar\psi(x)\gamma^\mu + M\bar\psi(x))\bar\pi_*(x)\Big).
\end{aligned}
\tag{7.14}
$$

As in the previous section, we quantize first and impose dynamics afterwards. However, this requires that $Q_{BRST}$ remains a well-defined, nilpotent operator even after normal ordering, which fails in the presence of fermions. In fact, even the gauge generators (7.13) fail to be well-defined. The dangerous part is $\mathcal{J}_X = -e \int d^N x \; X(x) :\psi(x)\pi(x):$. With $X = \exp(ik\cdot x)$, we have

$$
\begin{aligned}
\mathcal{J}(k) &= -e \int d^N x \; \mathrm{e}^{ik\cdot x} :\psi(x)\pi(x): \\
&= -e \int d^N k' \; :\psi(k')\pi(k-k'):.
\end{aligned}
\tag{7.15}
$$

One checks that

$$
[\mathcal{J}(k), \mathcal{J}(k')] = e^2 Z(k)\delta(k+k'), \tag{7.16}
$$

where

$$
Z(k) = \int_0^{k_0} dk'_0 \int_{-\infty}^{\infty} dk'_1 \ldots \int_{-\infty}^{\infty} dk'_{N-1} = k_0 \infty^{N-1}. \tag{7.17}
$$

The gauge algebra thus acquires a central extension. In one dimension, this is an affine algebra, but in more dimensions the central extension is infinite. So not only is the BRST operator not nilpotent, which we have to live with, but the anomalous gauge generators are not even operators, because infinities arise.

In this situation, the passage to jet space becomes indisposable, because it offers a way to regularize the theory so that the gauge generators become well defined. As usual we associate a family of jets to each field, e.g. $\psi(x) \to \psi_{,\mathbf{m}}(t)$. The action of the KT differential in jet space is given by

$$\begin{aligned}
\delta\psi_{,\mathbf{m}}(t) &= 0, \\
\delta\bar\psi_{,\mathbf{m}}(t) &= 0, \\
\delta\psi^*_{,\mathbf{m}}(t) &= \sum_\mu i\gamma^\mu\psi_{,\mathbf{m}+\mu}(t) - e\sum_\mu \gamma^\mu(A_\mu\psi)_{,\mathbf{m}}(t) - M\psi_{,\mathbf{m}}(t), \\
\delta\bar\psi^*_{,\mathbf{m}}(t) &= \sum_\mu i\bar\psi_{,\mathbf{m}+\mu}(t)\gamma^\mu + e\sum_\mu (A_\mu\bar\psi)_{,\mathbf{m}}(t)\gamma^\mu + M\bar\psi_{,\mathbf{m}}(t), \\
\delta A_{\mu,\mathbf{m}}(t) &= 0, \\
\delta A^\mu_{*,\mathbf{m}}(t) &= \sum_\nu F^{\mu\nu}_{,\mathbf{m}+\nu}(t) - j^\mu_{,\mathbf{m}}(t), \\
\delta\zeta_{,\mathbf{m}}(t) &= \sum_\mu A^\mu_{*,\mathbf{m}+\mu}(t) - ie(\bar\psi^*\psi)_{,\mathbf{m}}(t) - ie(\bar\psi\psi^*)_{,\mathbf{m}}(t),
\end{aligned} \tag{7.18}$$

where

$$\begin{aligned}
F_{\mu\nu,\mathbf{m}}(t) &= A_{\nu,\mathbf{m}+\mu}(t) - A_{\mu,\mathbf{m}+\nu}(t), \\
j^\mu_{,\mathbf{m}}(t) &= e(\bar\psi\gamma^\mu\psi)_{,\mathbf{m}}(t).
\end{aligned} \tag{7.19}$$

We also introduce extra antifields to cancel the $t$ dependence, but they will not be explicitly described here. Moreover, we introduce jet momenta, e.g. $\pi^{,\mathbf{m}}(t) = \delta/\delta\psi_{,\mathbf{m}}(t)$.

At this point we regularize the theory; simply ignore the jets for the original fields with $|\mathbf{m}| > p$. Since the Dirac equation is first order, Maxwell's equations are second order, and the continuity equation is third order, we truncate the jets as follows

| afn | Jet | Momentum | Order |
|---|---|---|---|
| 0 | $\psi_{,\mathbf{m}}(t), \bar\psi_{,\mathbf{m}}(t)$ | $\pi^{,\mathbf{m}}(t), \bar\pi^{,\mathbf{m}}(t)$ | $p$ |
| 0 | $A_{\mu,\mathbf{m}}(t)$ | $E^{\mu,\mathbf{m}}(t)$ | $p$ |
| 1 | $\psi^*_{,\mathbf{m}}(t), \bar\psi^*_{,\mathbf{m}}(t)$ | $\pi_*^{,\mathbf{m}}(t), \bar\pi_*^{,\mathbf{m}}(t)$ | $p-1$ |
| 1 | $A^\mu_{*,\mathbf{m}}(t)$ | $E_\mu^{*,\mathbf{m}}(t)$ | $p-2$ |
| 2 | $\zeta_{,\mathbf{m}}(t)$ | $\chi^{,\mathbf{m}}(t)$ | $p-3$ |

(7.20)

The KT operator reads explicitly

$$
\begin{aligned}
Q_{KT} \quad &= \quad \int dt\, \Big\{ \sum_{|\mathbf{m}|\leqslant p-2} (\sum_{\mu\nu} F^{\mu\nu}_{,\mathbf{m}+\nu}(t) E^{*,\mathbf{m}}_{\mu}(t) - \sum_{\mu} j^{\mu}_{,\mathbf{m}}(t) E^{*,\mathbf{m}}_{\mu}(t)) \\
&+ \quad \sum_{|\mathbf{m}|\leqslant p-3} (\sum_{\mu} A^{\mu}_{*,\mathbf{m}+\mu}(t) - ie(\bar\psi^{*}\psi)_{,\mathbf{m}}(t) - ie(\bar\psi\psi^{*})_{,\mathbf{m}}(t))\chi^{,\mathbf{m}}(t) \\
&+ \quad \sum_{|\mathbf{m}|\leqslant p-1} (\sum_{\mu} \gamma^{\mu}(i\psi_{,\mathbf{m}+\mu}(t) - e(A_{\mu}\psi)_{,\mathbf{m}}(t)) - M\psi_{,\mathbf{m}}(t))\pi^{,\mathbf{m}}_{*}(t) \\
&+ \quad \sum_{|\mathbf{m}|\leqslant p-1} (\sum_{\mu} (i\bar\psi_{,\mathbf{m}+\mu}(t) + e(A_{\mu}\bar\psi)_{,\mathbf{m}}(t))\gamma^{\mu} + M\bar\psi_{,\mathbf{m}}(t))\bar\pi^{,\mathbf{m}}_{*}(t) \Big\}.
\end{aligned}
\tag{7.21}
$$

The gauge generators (7.13) are truncated to

$$
\begin{aligned}
\mathcal{J}_X \quad &= \quad \int dt\, \Big\{ \sum_{|\mathbf{m}|\leqslant p} \sum_{\mu} \partial_{\mathbf{m}+\mu} X(q(t)) E^{\mu,\mathbf{m}}(t) \qquad (7.22) \\
&+ \quad e \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p} \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X(q(t)) (\, :\bar\psi_{,\mathbf{n}}(t)\bar\pi^{,\mathbf{m}}(t): \, - \, :\psi_{,\mathbf{n}}(t)\pi^{,\mathbf{m}}(t): ) \\
&+ \quad e \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p-1} \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X(q(t)) (\, :\bar\psi^{*}_{,\mathbf{n}}(t)\bar\pi^{,\mathbf{m}}_{*}(t): \, - \, :\psi^{*}_{,\mathbf{n}}(t)\pi^{,\mathbf{m}}_{*}(t): ) \Big\}.
\end{aligned}
$$

In particular, the dangerous part

$$
\mathcal{J}_X = -e \int dt \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p} \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X(q(t)) :\psi_{,\mathbf{n}}(t)\pi^{,\mathbf{m}}(t): \tag{7.23}
$$

now satisfies the algebra

$$
\begin{aligned}
[\mathcal{J}_X, \mathcal{J}_Y] \quad &= \quad \frac{k}{2\pi i} \int dt\ \dot X(q(t)) Y(q(t)) \\
&= \quad \frac{k}{2\pi i} \int dt\ \dot q^{\mu}(t) \partial_{\mu} X(q(t)) Y(q(t)),
\end{aligned}
\tag{7.24}
$$

where the “abelian charge” is

$$
k = e^2 n \binom{N+p}{N}, \tag{7.25}
$$

and $n = 2^{[N/2]}$ is the number of spinor components. The name “abelian charge” for the parameter in an abelian extension was introduced in [9] in

analogy with central charge for central extensions. The extension (7.24) is in fact central if only the gauge generators are taken into account, but it does not commute with arbitrary diffeomorphisms.

The regularized theory can now be quantized. We expand all jets ($\psi_{,\mathbf{m}}(t)$ etc.) in a Fourier series in $t$ and introduce a vacuum $\big|0\big\rangle$ which is annihilated by all negative frequency modes. The KT operator (7.21) is already normal ordered and thus a well-defined operator, so we can introduce dynamics by passing to the KT cohomology. Moreover, the theory so defined is a gauge theory on the quantum level, because the gauge generators (7.22) act in a well-defined manner on the history Hilbert space, and they commute with the KT operator (7.21):

$$
[\mathcal{J}_X, Q_{KT}] = 0. \tag{7.26}
$$

However, making the gauge symmetry well-defined on the quantum level comes with a price: the term on the RHS of (7.24) is a anomaly. This means that the quantum BRST charge (7.14) is no longer nilpotent. To understand the physical implications of this, let us go back to the classical level, and consider the action of the KT differential on the Fourier modes of the fields.

$$
\begin{aligned}
\delta\psi(k) &= 0, \\
\delta\bar\psi(k) &= 0, \\
\delta\psi^*(k) &= (k_\mu\gamma^\mu - M)\psi(k) - e\gamma^\mu \int d^N k' A_\mu(k-k')\psi(k'), \\
\delta\bar\psi^*(k) &= \bar\psi(k)(k_\mu\gamma^\mu + M) + e\int d^N k' A_\mu(k-k')\bar\psi(k')\gamma^\mu, \\
\delta A_\mu(k) &= 0, \\
\delta A_*^\mu(k) &= k^\mu k_\nu A^\nu(k) - k^\nu k_\nu A^\mu(k) + j^\mu(k), \\
\delta\zeta(k) &= k_\mu A_*^\mu(k) - ie\int d^N k' (\bar\psi^*(k-k')\psi(k') + \bar\psi(k-k')\psi^*(k')).
\end{aligned} \tag{7.27}
$$

If we ignore the non-linear terms proportional to $e$, the physical phase space is spanned by fermions satisfying the Dirac equation and massless photons. Unlike the free Maxwell case, however, there are no ghosts $c$, and therefore there is one additional physical photon polarization. To see this in detail, let $k^2 = 0$, say $k = (k_0, 0, 0, k_0)$. Then $sA_\mu = 0$, $sA_*^0 = sA_*^3 = k^0 k_\nu A^\nu$, $sA_*^1 = sA_*^2 = 0$ and $s\zeta = k_\mu A_*^\mu$. The kernel is thus spanned by $A_\mu$, $A_*^1$, $A_*^2$, and $k_\mu A_*^\mu = A_*^0 - A_*^3$. The image is spanned by $k^\mu A_\mu = A_0 - A_3$ and $k_\mu A_*^\mu$, which factor out in cohomology. We are left with three physical polarizations $A_1$, $A_2$, and $A_0 + A_3$.

Three photon polarizations might seem counter-intuitive, and one might wonder what happens with unitarity in this situation. This question is beyond the scope of the present paper, since we have not even defined the Hilbert space inner product. However, what must be remembered is that we are dealing with the Hilbert space of the full, interacting theory, and hence the photons are virtual. It is perfectly legitimate for virtual photons to have unphysical polarizations, and this is a necessary price for having a well-defined gauge symmetry on the quantum level. The unphysical polarizations must vanish in regions of spacetime where there is no charge. However, in that situation we are dealing with a free Maxwell field, and the analysis in the previous section applies.

Finally, we must address the question of removing the regularization, i.e. to take the limit $p \to \infty$. This limit is problematic, because the abelian charge (7.25) diverges in all dimensions $N > 0$. This is perhaps not so surprising. We had to reject the field formulation because the central extension in (7.16) was infinite. An infinite jet is essentially the same thing as the field itself, and a physical divergence must therefore resurface in the jet formalism.

However, the situation is not quite as bad as it seems. The antifields also contribute to the abelian charge, but with opposite sign because they have opposite Grassmann parity. Since the Dirac equation is first order, the antifields are truncated at order $p-1$, so the total abelian charge becomes

$$k = e^2 n \binom{N+p}{N} - e^2 n \binom{N+p-1}{N} = e^2 n \binom{N-1+p}{N-1}. \tag{7.28}$$

This expression still diverges when $p \to \infty$ in all dimensions $N > 1$. Moreover, recall that we implicitly need further antifields to cancel the $t$ dependence as in (4.3). After taking these extra antifields, which are truncated at one order lower than the original fields and have opposite parity, the abelian charge becomes

$$k = e^2 n \binom{N+p-1}{N-1} - e^2 n \binom{N+p-2}{N-1} = e^2 n \binom{N-2+p}{N-2}, \tag{7.29}$$

which has a finite limit when $N \leqslant 2$. For more complicated theories further cancellations may be possible, pushing up the critical dimension.

It is important to realize that Maxwell's equations (7.3) and the conti-

nuity equation (7.5), or rather their jet counterparts,

$$
\begin{aligned}
&\sum_\nu (F^{\mu\nu}_{,\mathbf{m}+\nu}(t) - j^\mu_{,\mathbf{m}}(t)) = 0, \\
&\sum_\mu j^\mu_{,\mathbf{m}+\mu}(t) \equiv 0,
\end{aligned}
\tag{7.30}
$$

are realized as operator equations in the KT cohomology. This is ensured by the fact that the KT operator remains anomaly free after quantization, unlike the BRST operator. Hence the extension (7.24) of the algebra of gauge transformations does not ruin charge conservation. This is possible because we do not identify the momentum $\pi(x)$ with the Dirac conjugate $\bar\psi(x)$, and hence the current (7.4) is distinct from the gauge generators (7.13).

In fact, without the anomaly it is impossible to write down a well-defined operator for electic charge. By definition, the electric charge operator $Q_{el}$ satisfies

$$
\begin{aligned}
&[Q_{el}, \psi(x)] = -e\psi(x), \qquad [Q_{el}, \bar\psi(x)] = e\bar\psi(x), \\
&[Q_{el}, A_\mu(x)] = 0.
\end{aligned}
\tag{7.31}
$$

In jet space, this becomes

$$
\begin{aligned}
&[Q_{el}, \psi_{,\mathbf{m}}(t)] = -e\psi_{,\mathbf{m}}(t), \qquad [Q_{el}, \bar\psi_{,\mathbf{m}}(t)] = e\bar\psi_{,\mathbf{m}}(t), \\
&[Q_{el}, A_{\mu,\mathbf{m}}(t)] = 0.
\end{aligned}
\tag{7.32}
$$

Ignoring the action on the antifields, electric charge is thus measured by the operator

$$
Q_{el} = e \sum_{|\mathbf{m}|\leqslant p} \int dt\, \Big( {:}\bar\psi_{,\mathbf{m}}(t)\bar\pi^{,\mathbf{m}}(t){:} \,-\, {:}\psi_{,\mathbf{m}}(t)\pi^{,\mathbf{m}}(t){:} \Big). \tag{7.33}
$$

This is recognized as the special gauge transformation (7.22) $Q_{el} = \mathcal{J}_X$ with $X(x) \equiv 1$.

Since the electric charge is a gauge generator, it would be impossible to have non-zero electric charge in the BRST cohomology. In the KT cohomology, on the other hand, electric charge is a conventional symmetry generator, because of the anomaly. It is easy to see that unitarity in fact requires the extension in (7.24). Namely, consider the restriction to the subalgebra generated by $\mathcal{J}_X$ where $X(x) = X(x^0, 0, ..., 0)$ only depends on the $x^0$ coordinate. This subalgebra is identified with the affine algebra $\widehat{\mathfrak{u}(1)}$. It is well known that affine algebras have non-trivial unitary representations only when the central charge is positive [4]. Since $Q_{el} \neq 0$ means that the representation is non-trivial, unitarity requires a non-zero anomaly.

# 8 Yang-Mills theory

Non-abelian Yang-Mills theories are structurally quite similar to the Maxwell theory studied in the previous sections, although the expressions are more cumbersome. The main difference from our point of view is that the gauge algebra becomes anomalous already for the pure theory, because the quadratic Casimir no longer vanishes in the adjoint representation. In other words, already pure Yang-Mills theory is an interacting theory.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with structure constants $f_{ab}{}^c$ and Killing metric $\delta^{ab}$, which we freely use to raise and lower $\mathfrak{g}$ indices. The gauge potential is denoted by $A^a_\mu(x)$ and the exterior derivative reads

$$\mathcal{D}_\mu = \partial_\mu + iA^a_\mu(x)T_a, \tag{8.1}$$

where the matrices $T_a$ belong to some finite-dimensional representation of $\mathfrak{g}$. The field strength is

$$F_{\mu\nu}(x) = [\mathcal{D}_\mu, \mathcal{D}_\nu] = F^a_{\mu\nu}(x)T_a. \tag{8.2}$$

The action

$$S = \frac{1}{4}\int d^N x\ F^a_{\mu\nu}(x)F^{\mu\nu}_a(x) \tag{8.3}$$

leads to the Yang-Mills equations of motion,

$$\mathcal{D}_\nu F^{\mu\nu}_a(x) = 0. \tag{8.4}$$

We introduce the KT and BRST differentials; the latter reads

$$\begin{aligned}
sc^a(x) &= -\frac{1}{2}f_{bc}{}^a c^b(x)c^c(x), \\
sA^a_\mu(x) &= \mathcal{D}_\mu c^a(x), \\
sA^\mu_{*a}(x) &= \mathcal{D}_\nu F^{\mu\nu}_a(x) + f_{ab}{}^c A^\mu_{*c}(x)c^b(x), \\
s\zeta_a(x) &= \mathcal{D}_\mu A^\mu_{*a}(x) + f_{ab}{}^c \zeta_c(x)c^b(x),
\end{aligned} \tag{8.5}$$

where

$$\mathcal{D}_\mu c^a(x) = \partial_\mu c^a - f_{bc}{}^a A^b_\mu(x)c^c(x), \tag{8.6}$$

in agreement with (8.1) After passage to jet space, the KT differential becomes

$$\begin{aligned}
\delta A^a_{\mu,\mathbf{m}}(t) &= 0, \\
\delta A^\mu_{*a,\mathbf{m}}(t) &= (\mathcal{D}_\nu F^{\mu\nu}_a)_{,\mathbf{m}}(t), \\
\delta\zeta_{a,\mathbf{m}}(t) &= (\mathcal{D}_\mu A^\mu_{*a})_{,\mathbf{m}}(t).
\end{aligned} \tag{8.7}$$

The gauge generators

$$
\begin{aligned}
\mathcal{J}_X &= \int dt\, \Big\{ \Big( \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p} \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X^a(q(t)) f_{ab}{}^c {:} A^b_{\mu,\mathbf{n}}(t) E^{\mu,\mathbf{m}}_c(t){:} \\
&\qquad + \sum_{|\mathbf{m}|\leqslant p} \sum_\mu \partial_{\mathbf{m}+\mu} X^a(q(t)) E^{\mu,\mathbf{m}}_a(t) \Big) \\
&\quad - \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p-2} \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X^a(q(t)) f_{ab}{}^c {:} A^\mu_{*c,\mathbf{n}}(t) E^{*b,\mathbf{m}}_\mu(t){:} \\
&\quad - \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p-3} \binom{\mathbf{m}}{\mathbf{n}} \partial_{\mathbf{m}-\mathbf{n}} X^a(q(t)) f_{ab}{}^c {:} \zeta_{c,\mathbf{n}}(t) \chi^{b,\mathbf{m}}(t){:} \Big\}
\end{aligned}
\tag{8.8}
$$

satisfy the algebra

$$
\begin{aligned}
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_{[X,Y]} + \frac{k}{2\pi i} \int dt\ \dot{X}_a(q(t)) Y^a(q(t)) \\
&= \mathcal{J}_{[X,Y]} + \frac{k}{2\pi i} \delta^{ab} \int dt\ \dot{q}^\mu(t) \partial_\mu X_a(q(t)) Y_b(q(t)),
\end{aligned}
\tag{8.9}
$$

where the abelian charge has three contributions from the three sums in (8.8):

$$
k = f_{ac}{}^d f_{bd}{}^c \Big( \binom{N+p}{N} - \binom{N+p-2}{N} + \binom{N+p-3}{N} \Big). \tag{8.10}
$$

$k \neq 0$ because the quadratic Casimir $f_{ac}{}^d f_{bd}{}^c > 0$, so there is an anomaly in the gauge algebra, which does not cancel the contribution from the ghost $c^a_{,\mathbf{m}}(t)$. The BRST operator is not nilpotent and only the KT cohomology can be implemented. In four dimensions, we have transverse gluons $A^a_1$ and $A^a_2$, but also the polarization $A^a_0 + A^a_3$ survives. This is not a contradiction, because nonabelian Yang-Mills theory is an interacting theory already without fermions, and virtual gluons need not be transverse. The passage to $p$-jet space is a regularization which can not be removed. The leading divergences in the $p \to \infty$ limit of (8.10) do not cancel; that would require a term proportional to $\binom{N+p-1}{N}$.

We may ask under which condition the abelian charge $k$ has a finite $p \to \infty$ limit in four dimensions. To that end, consider Yang-Mills theory coupled to a fermionic spinor field $\psi$. Let $x_B$ and $x_F$ denote the values of the quadratic Casimir for the bosonic and fermionic fields, i.e. tr $T_a T_b = x\delta_{ab}$,

and let $n_F$ denote the number of fermionic species. As in [10], we find that the contributions from the fields and the various antifields are

$$
\begin{array}{|c|c|c|l|}
\hline
\text{afn} & \text{Jet} & \text{Order} & x \\
\hline
0 & \psi_{,\mathbf{m}}(t) & p & n_F x_F \\
1 & \psi^*_{,\mathbf{m}}(t) & p-1 & -n_F x_F \\
\hline
\hline
0 & A^a_{\mu,\mathbf{m}}(t) & p & -x_B \\
1 & A^\mu_{*a,\mathbf{m}}(t) & p-2 & x_B \\
2 & \zeta_{a,\mathbf{m}}(t) & p-3 & -x_B \\
\hline
\end{array}
\tag{8.11}
$$

In order for the abelian charges to have a finite $p \to \infty$ limit, the contributions at order $p - r$ in $N$ dimensions must be

$$
x_r = (-1)^r \binom{N}{r} X. \tag{8.12}
$$

However, there are also antifields which cancel to $t$-dependence as in (4.3), so the above equation is replaced by

$$
x_r = (-1)^r \binom{N-1}{r} X. \tag{8.13}
$$

In particular, if we put $N = 4$ and substitute the contributions from (8.11), we find

$$
\begin{aligned}
p: &\qquad n_F x_F - x_B = X \\
p-1: &\qquad -n_F x_F = -3X, \\
p-2: &\qquad x_B = 3X, \\
p-3: &\qquad -x_B = -X.
\end{aligned}
\tag{8.14}
$$

These conditions clearly have no non-trivial solutions.

# 9 Conclusion

In this paper we have extended the manifestly covariant canonical quantization method, introduced in [12], to theories with gauge symmetries. The method is exact but implicit. We describe the regularized Hilbert spaces exactly as cohomology spaces, but in order to extract numbers we need a more explicit description. This presumably requires the introduction of perturbation theory and renormalization into this formalism.

The passage to $p$-jets is a regularization, and in the end we must remove the regularization by taking the limit $p \to \infty$. It is clearly necessary that all abelian charges remain finite in this limit. It was hoped in [10, 11] that the leading infinities would cancel for some particular field content describing our world. Under reasonable assumptions, this requirement uniquely fixed spacetime dimension to $N = 4$, and vaguely suggested three generations of quarks and leptons, but the details do not seem to work out. In particular, it is disappointing that the conditions (8.14) lack solutions. Perhaps this hints that further renormalization is necessary.

To quantize regularized field theories might not seem overly impressive. However, it should be emphasized that also the regularized theories carry representations of the *full* constraint algebras, in the right number of dimensions, and that this feature makes Taylor-expansion regularization unique. It is sometimes claimed that lattice gauge theory implements the gauge group exactly, but this is not quite true. The lattice gauge group is the group $Map(\Lambda, G)$ of maps from the finite lattice $\Lambda$ into $G$, and this group is different from the continuum gauge group $Map(\mathbb{R}^N, G)$. In particular, $Map(\Lambda, G)$ is finite-dimensional, so there are no gauge anomalies.

The main novelty is the appearence of new gauge anomalies. This phenomenon is genuinely new, because neither a Yang-Mills anomaly proportional to the quadratic Casimir, nor a pure diffeomorphism anomaly in four dimensions, can arise in conventional quantum field theory [1]. The anomaly vanishes in the particular case of the free Maxwell field, because the second Casimir is zero in the adjoint representation of $\mathfrak{u}(1)$, but it is generically present in interacting theories.

Describing physics as cohomology in the history phase space is very convenient, because it maintains manifest covariance at all times. Indeed, the history phase space is the natural habitat for the algebras of gauge transformations and diffeomorphisms. However, it is presumably possible to repeat the analysis, including the anomalies, using a privileged foliation of spacetime. In constrast, to introduce the observer's trajectory is essential, because otherwise it is impossible to even write down the new, observer-dependent anomalies.

We also argued that these new anomalies do not imply any inconsistency, unlike conventional gauge anomalies caused by chiral fermions coupled to gauge fields. On the contrary, consistency (unitarity) is guaranteed by unitary representations, and all non-trivial unitary irreps of gauge and diffeomorphism algebras are projective. In particular, unitary time evolution is guaranteed if the Hamiltonian is an element in a unitarily represented algebra, anomalous or not. Moreover, a non-zero electric charge (7.33), which

certainly is a necessary physical requirement, can only be a well-defined operator in the presence of the anomaly. It would be inconsistent, however, to try to treat an anomalous gauge symmetry as a gauge symmetry. Rather, the anomaly makes classical gauge degrees of freedom physical on the quantum level, turning the classical gauge symmetry into a conventional global quantum symmetry. The physical consequence is typically that virtual photons and gluons may have unphysical polarizations.

Despite the phenomenological success of quantum field theory, these new anomalies can not be ignored in a final theory, simply because they exist mathematically. Anomalies matter!

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